Study Notes for Monotone Comparative Statics

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Contents

Chapter 1

Monotone Comparative Statics

1.1 Summary

1. For a single function, Theorem 4 of (Milgrom and Shannon 1994) shows the necessary and sufficient condition for monotone comparative statics: Let $f : X \times T \to R$, where X is a lattice, T is a partially ordered set and $S \subset X$. Then $\arg\max_{x\in S} f(x,t)$ is monotone nondecreasing in (t, S) if and only if f is quasisupermodular in x and satisfies the single crossing property in $(x; t)$.

2. With one dimensional action space, in order to have monotone comparative statics holds for single integration of function (summation of functions), Theorem 2 of (Quah and Strulovici 2010) establish one sufficient condition. In order to prove $\arg\max_{x\in R} \int_{\Theta_2} g(x,\theta_1,\theta_2)d\theta_2$ is monotone nondecreasing in (θ_1, R), let's define $f(\theta_1,\theta_2) = g(x'',\theta_1,\theta_2) - g(x',\theta_1,\theta_2)$, and we want to show $\int_{\Theta_2} f(\theta_1,\theta_2)d\theta_2$ satisfy SC1 (as the decision variable is one dimension, we do not need to consider the quasisupermodular in decision space). Theorem 2 of (Quah and Strulovici 2010) shows that: if $\{f(\cdot,\theta_2)\}_{\theta_2\in\Theta_2}$ an $\mathcal{S}$-summable family indexed by elements in Θ_2 and defined on Θ_1 ($f(\cdot,\theta_2') \sim f(\cdot,\theta_2'')$ for any $\theta_2',\theta_2'' \in \Theta_2$). For any fixed θ_1, $f(\theta_1,\cdot)$ is a measurable and bounded function of θ_2. Then the function $F : \Theta_1 \to \mathbb{R}$ defined by $F(\theta_1) = \int_{\Theta_2} f(\theta_1,\theta_2)\,d\theta_2$ is also an $\mathcal{S}$ function.

   For multiple integration cases, Theorem 3 of (Quah and Strulovici 2010) shows that: let $f : \Theta \to \mathbb{R}$ be a bounded and measurable $\mathbb{I}_1$ function. Then (i) $F_n : \Theta_{N_n} \to \mathbb{R}$ as defined by $F_n(\theta_{N_n}) = \int_{\Theta_n} f(\theta_{N_n},\theta_n)\,d\theta_n$ is an $\mathbb{I}_1$ and (ii) $F : \Theta_1 \to \mathbb{R}$ as defined by $F(\theta_1) = \int_{\Theta_2}\int_{\Theta_3}\cdots\int_{\Theta_n} f(\theta_1,\theta_2,...,\theta_{n-1},\theta_n)\,d\theta_n d\theta_{n-1}...d\theta$ is an $\mathcal{S}$ function.

3. For multi-dimensional decision variables case, in order to prove the monotone comparative statics for $U(X,\theta) = \int u(X,s)\,f(s,\theta)\,ds$, according to theorem 1 of (Athey 2002), one sufficient condition is that u is log-supermodular in (X,S) and f is log-supermodular in (S,θ).

1.2 Definitions and Functional properties

Definition 1 *((Milgrom and Shannon 1994)) Let X be a partially ordered set, with the transitive, reflexive, antisymmetric order relation $\geq$. The set X is a **lattice** if for every pair of elements x and y in X, the **joint** $x \vee y$, least upper bound if exists, and **meet** $x \wedge y$, the greatest lower bound if exists, do exist as elements of X.*

Definition 2 *((Milgrom and Shannon 1994)) A subset S of X is a **sublattice** of X if S is closed under the operations meet and join.*

Definition 3 *((Milgrom and Shannon 1994)) A sublattice S of X is **complete** if for every nonempty subset S' of S, $\inf(S')$ and $\sup(S')$ both exist and are elements of S.*

Definition 4 *((Veinott 1989))* ***Strong set order*** $\leq_S$*: For* X *a lattice with the given relation* $\geq$*, with* Y *and* Z *elements of the power set* $P(X)$*, we say that* $Z \leq_S Y$*, read "*Y *is higher than* Z*", if for every* $z \in Z$ *and* $y \in Y$*,* $z \vee y \in Y$ *and* $z \wedge y \in Z$*.*

Definition 5 *Given a partially ordered set* T*, we say that a set-valued function* $M : T \to P(X)$ *is* ***monotone nondecreasing*** *if* $t \leq t'$ *implies* $M(t) \leq_S M(t')$*. (In case* Z *and* Y *are singletons, then the strong set order* $\leq_S$ *coincides with the given order* $\leq$*on the underlying choice set.)*

Theorem 6 *((Topkis 1978))((Milgrom and Shannon 1994) Thm 1) A subset* S *of* R^n *is a sublattice if and only if there exist* $n(n-1)$ *functions* $g_{ij} : R^2 \to R (i \neq j)$*, each of which is increasing in its first argument and decreasing in its second, and* n *sets* $S_i \subset R (i = 1, ..., n)$ *such that* $S = \{x | g_{ij}(x_i, x_j) \leq 0 \text{ for all } 1 \leq ij \leq n\} \cap \{x | x_i \in S_i\}$*.*

Theorem 7 *((Milgrom and Shannon 1994) Thm 2)* S *and* S' *are complete sublattices of* X *such that* $S \geq_S S'$ *if and only if there exists a complete sublattice* R *of* X *such that* $S = R \cap \{x \geq \inf(S)\}$ *and* $S' = R \cap \{x \leq \sup(S')\}$*. Moreover, if* S *and* S' *are complete sublattices with* $S \geq_S S'$*, then* $S \cup S'$ *is a complete sublattice.*

Definition 8 *Let* X *be a lattice,* T *be a partially ordered set, and* $f : X \times T \to R$*. Then* f *satisfies the* ***single crossing property*** *in* $(x; t)$ *if for* $x' > x''$ *and* $t' > t''$*,* $f(x', t'') > f(x'', t'')$ *implies that* $f(x', t') > f(x'', t')$ *and* $f(x', t'') \geq f(x'', t'')$ *implies that* $f(x', t') \geq f(x'', t')$*. If* $f(x', t'') \geq f(x'', t'')$ *implies* $f(x', t') > f(x'', t')$*, then* f *satisfies the* ***strict single crossing property*** *in* $(x; t)$*.*

Remark 9 *Alternative definition ((Athey 2002)): For* $T \subseteq R$*, let* $g : T \to R$*.*

i. g *satisfies* ***single crossing function (SC1)*** *in* t *if there exists* $\inf T \leq t_0' \leq t_0'' \leq \sup T$ *such that* $g(t) < (\leq) 0$ *for all* $t < (\leq) t_0'$*,* $g(t) = 0$ *for all* $t_0' < t < t_0''$*, and* $g(t) > (\geq) 0$ *for all* $t > (\geq) t_0''$*. (*$g(t) > (\geq) 0$ *for all* $t > (\geq) t_0''$ *means that if* $t \geq t_0''$ *then* $g(t) \geq 0$ *and if* $t > t_0''$ *then* $g(t) > 0$*.)*

ii. $h : X \times T \to R$ *satisfies* ***single crossing properties*** *in two variables* ***(SC2)*** *in* (X, T) *if for all* $X_H > X_L$*,* $g(t) = h(X_H; t) - h(X_L; t)$ *satisfies SC1. (In (Quah and Strulovici 2010), SC2 is called single crossing difference and SC1 is called single crossing property)*

Remark 10 h *and* g *satisfy SC2 in* (X, T) *does NOT mean* $h + g$ *satisfies SC2 in* (X, T)*.(Check (Quah and Strulovici 2010) Prop 1)*

Definition 11 *We say that* $g : R \to R$ *satisfies* ***weak SC1*** *if there exists a* t_0 *such that* $g(t) \leq 0$ *for all* $t < t_0$ *and* $g(t) \geq 0$ *for all* $t > t_0$*.*

Definition 12 *We say that* $h : X \times R \to R$ *satisfies* ***weak SC2*** *in* $(X; t)$ *if for all* $X_H > X_L$*,* $g(t) = h(X_H; t) - h(X_L; t)$ *satisfies weak SC1.*

Definition 13 *A decision maker of type* t *who chooses a point* $(x, y) \in R^2$ *has a payoff of* $U(x, y; t)$*. A continuously differentiable function* U *on a rectangular domain with* $U_y \neq 0$ *satisfies the* ***(strict) Spence-Mirrlees condition (SM)*** *if* $U_x / |U_y|$ *is nondecreasing (increasing) in* t *for any fixed* (x, y)*.*

Theorem 14 *((Milgrom and Shannon 1994) Thm 3) Let* R^2 *be given the lexicographic order, with* $(x, y) \geq (x', y')$ *if either* $x > x'$ *or* $x = x'$ *and* $y \geq y'$*. Suppose that* $U(x, y, t) : R^3 \to R$ *is completely regular and twice continuously differentiable with* $U_y \neq 0$*. Then* $U(x, y; t)$ *has the (strict) single crossing property in* $(x, y; t)$ *if and only if it satisfies the (strict) Spence-Mirrlees condition.*

Remark 15 *When choice set is totally ordered (e.g.* R*), the single crossing property is the only condition we will need for comparative statics. However, when the choice set is not totally ordered (e.g.* R^2*), an additional condition (quasisupermodularity) is necessary.*

Definition 16 *Given a lattice* X*, we say that a function* $f : X \to R$ *is* ***quasisupermodular*** *if (i)* $f(x) \geq f(x \wedge y)$ *implies* $f(x \vee y) \geq f(y)$ *and (ii)* $f(x) > f(x \wedge y)$ *implies* $f(x \vee y) > f(y)$ *.*

Lemma 17 *It is equivalent to say that $f : X \to R$ is quasisupermodular if and only if it satisfies SC2 in $(x_i; x_j)$ for all $i \neq j$. (Note that SC2 in $(x_i; x_j)$ for all $i \neq j$ means must satisfy $(x_1; x_2)$ and $(x_2; x_1)$, because SC2 in $(x_1; x_2)$ does not mean SC2 in $(x_2; x_1)$.) (Hence, quasisupermodular seems more strong than SC2.)*

Remark 18 *When $X = R$ so that the choice set is totally ordered, every function is quasisupermodular, as the order operations meet and join are then trivial. When $X = R^2$, requiring f to be quasisupermodular is equivalent to requiring that f satisfy the single crossing property in $(x_1; x_2)$ and also in $(x_2; x_1)$.*

Essentially, quasisupermodularity expresses a weak kind of complementarity between the choice variables: if an increase in some subset of the choice variables is desirable at some level of the remaining choice variables, it will remain desirable as the remaining variables also increases.

Corollary 19 *$f(x)$ is quasisupermodular if and only if $\arg\max_{x \in S} f(x)$ is monotone nondecreasing in S.*

Corollary 20 *If S is a sublattice of X, and f is quasisupermodular, then $\arg\max_{x \in S} f(x,t)$ is a sublattice of S.*

Definition 21 *A function $f : R^m \to R$ is **supermodular** if, let all $x, y \in X$ and X is lattice,*

$$f(x \wedge y) + f(x \vee y) \geq f(x) + f(y)$$

where $x \wedge y = (\min(x_1, y_1), ..., \min(x_m, y_m))$; $x \vee y = (\max(x_1, y_1), ..., \max(x_m, y_m))$

(The test of supermodularity for a multidimensional function f can be reduced to test the pairwise supermodularity of f. ((Athey 1998)))

Theorem 22 *(Theorem 1 of (Nachbar 2009)) The following are equivalent:*

1. f is supermodular;
2. $-f$ is submodular;
3.Let $f : R^m \to R$, $f(x \wedge y) + f(x \vee y) \geq f(x) + f(y)$ for all $x, y \in X$ and X is lattice;
4. Let $f : R^2 \to R$, $f(x^*, \theta^*) - f(x^o, \theta^*) \geq f(x^*, \theta^o) - f(x^o, \theta^o)$ for all $x^o \leq x^*$ and $\theta^o \leq \theta^*$;
5. Let $f : R^2 \to R$ be C^1, $D_x f(x^*, \theta^*)$ is weakly increasing in θ, for every x^*; (If)
6. Let $f : R^2 \to R$ be C^1, $D_\theta f(x^*, \theta^*)$ is weakly increasing in x, for every θ^*;
7. Let $f : R^2 \to R$ be C^2, $D^2_{\theta,x} f(x^*, \theta^*) \geq 0$ for every (x^*, θ^*);

Definition 23 *h is **log-supermodular** if it is non-negative and for all $x, y \in X$, $h(x \wedge y) \cdot h(x \vee y) \geq h(x) \cdot h(y)$.*

(The test of log-supermodularity for a multidimensional function f can be reduced to test the pairwise log-supermodularity of f given f is positive. ((Athey 1998)))

(The test of log-supermodularity for a multidimensional function f can be tested by whether it holds "one dimension at a time". I.e. a function g is logsupermodular if and only if $g(\cdot, s''_K)/g(\cdot, s'_K)$ is increasing in the scalar $s_{N \setminus K}$ whenever $s''_K > s'_K$, where K has exactly $n-1$ elements.((Quah and Strulovici 2010)))

Theorem 24 *When the support of $F(\cdot; \theta)$ is constant in θ, and F has a density f, then the Monotone Likelihood Ratio Order requires that f is log-supermodular, that is, the likelihood ratio $f(s; \theta_H)/f(s; \theta_L)$ is nondecreasing in s for all $\theta_H > \theta_L$.*

Theorem 25 *((Athey 2002)) If h is positive, then h is log-supermodular if and only if $\log h(\cdot)$ is supermodular.*

Remark 26 *Supermodularity means that increasing any subset of the decision variables raises the incremental returns associated with increases in the others.*

Supermodular and log-supermodular functions are stronger property than quasisupermodular. (Supermodular and log-supermodular functions are subset of quasisupermodular function.)

Supermodular and log-supermodular are stronger property than SC2.

Theorem 27 *Preservation property of log-supermodularity*

1. Product of log-supermodular functions are log-supermodular function,

2. If $h(X,\theta)$ is log-supermodular in ALL arguments, then log-supermodularity is preserved under summation and integration; (By (Ahlswede and Daykn 1978)) (the sum of log-supermodular functions are not necessary log-supermodular function. E.g. if $h(X,\theta)$ is log-supermodular only on X, then $\int h(X,\theta)\,d\theta$ is not necessary log supermodular on X.)

Corollary 28 *If a function is supermodular then it is also quasisupermodular.*

Definition 29 *The function $f : X \times T \to R$ has **increasing differences** in (x,t) if for $x' \geq x$, $f(x',t) - f(x,t)$ is monotone nondecreasing in t.*

Remark 30 *Increasing differences means that increasing a parameter raises the marginal return to activities.*

If a function has increasing differences in (x,t), then it also satisfy the single crossing property in (x,t) as well.

Theorem 31 *Supermodularity is preserved under maximization, limits, addition, expectation, and integration*

1. Non-negative weighted maximum: $f = \max\{w_1 f_1, ..., w_n f_n\}$ is supermodular, if $f_1, ..., f_n$ are supermodular; $w_1, ..., w_n$ are non-negative;

2. Limitation operation: if f^n are supermodular, then $\lim_{n\to\infty} f^n$ is supermodular;

3. Summation: if f and g are supermodular, then $f + g$ is supermodular; (so supermodular is preserved under expectation and integration) ((Vives 1990))

4. Scalar: if f is supermodular and s is a scalar, then sf is supermodular; (Lemma 8.3 of (Evan L. Porteus 2002))

5. Loss function: if f is supermodular, then $h(x) = E[f(x - D)]$ is supermodular, where D is a r.v.; (Lemma 8.3 of (Evan L. Porteus 2002));

6. Maximization: If $(s,x) \in C$ is lattice and $s \in S$ is lattice. Let $f(s) = \max_{x\in X} g(s,x)$ and $g(s,x)$ is supermodular for $(s,x) \in C$. Then f is supermodular on S. (Similarly, submodularity is preserved under minimization.)

7. If f is supermodular and increasing and $g : R \to R$ is increasing and convex, then $g \circ f$ is supermodular. ((Milgrom and Shannon 1994) Theorem 7)

(submodular is preserved under minimization, limits, addition, expectation, and integration)

Theorem 32 *For any quasisupermodular function $f : X \to R$ and any strictly increasing function $g : R \to R$, the composition $g(f(x))$ is also quasisupermodular.*

If f is supermodular, then $g(f(x))$ is quasisupermodular for any strictly increasing function g.

If there exists some strictly increasing function $h : R \to R$ such that $h(f(x))$ is supermodular, then the original function f is quasisupermodular.

Definition 33 *((Calzi 1990) Generalized symmetric supermodular functions) If there exists some increasing function $h : R \to R$ such that $h(f(x))$ is (strictly) supermodular, then f function is called **(Strictly) supermodularizable**. (From the above theorem, we know that supermodularizable function is a subset of quasisupermodular function. Supermodular function and supermodularizable function have intersection with in quasi-supermodular function.)*

Lemma 34 *((Milgrom and Shannon 1994)) Suppose $f : X \to R$ is quasisupermodular, where $X = \{x, x', x \vee x', x \wedge x'\}$, then there exists some $h : R \to R$ such that h is strictly increasing and $h \circ f : X \to R$ is supermodular. (Note that the domain of f is restricted to four elements.)*

Theorem 35 *((Milgrom and Shannon 1994) Thm 8) Let X be a lattice and $f : X \to R$. Then f is quasisupermodular if and only if there exists some $g : R \times X \times X \to R$ such that (i) $g(r, x_1, x_2)$ is strictly increasing in r for every fixed $(x_1, x_2) \in X^2$ and (ii) for every $x_1, x_2 \in X$, $g(f(x), x_1, x_2)$ is supermodular in x on the sublattice $\{x_1, x_2, x_1 \vee x_2, x_1 \wedge x_2\}$.*

Theorem 36 *((Milgrom and Shannon 1994) Thm 9) Let X be a lattice and T a partially ordered, finite set and $f : X \times T \to R$. Then f has the single crossing property if and only if there exists $g : R \times X^2 \times T \to R$ such that g is increasing in its first argument and for all $x_1 \geq x_2$ in X, $g(g(x_1, x_2), x_1, x_2, t)$ has increasing differences in $(x; t)$ on the set $\{x_1, x_2\} \times T$.*

1.3 Comparative statics under certainty

Theorem 37 *(Monotonicity Theorem in (Milgrom and Shannon 1994) Thm 4): Let $f : X \times T \to R$, where X is a lattice, T is a partially ordered set and $S \subset X$. Then $\arg\max_{x \in S} f(x, t)$ is monotone nondecreasing in (t, S) if and only if f is quasisupermodular in x and satisfies the single crossing property in $(x; t)$.*

Theorem 38 *((Milgrom and Shannon 1994) Thm 5)Let X be a lattice, T a partially ordered set, and $f : X \times T \to R$. If $f(x, t)$ is supermodular in x and has increasing differences in $(x; t)$, then $\arg\max_{x \in S} f(x, t)$ is monotone nondecreasing in (t, S).*

Theorem 39 *(Combine (Milgrom and Shannon 1994) Thm 8&9) Given a function $f : X \times T \to R$, if there exists a function $g : R \times T \to R$ such that g is increasing in its first argument for every t and such that $g(f(x, t), t)$ is supermodular in x and has increasing difference in (x, t), then f if quasisupermodular in x and satisfies the single crossing property in (x, t).*

Theorem 40 *((Milgrom and Shannon 1994)) Let $f : R^n \times R \to R$. Then $f(x, t) + p \cdot x$ is quasisupermodular in x and has the single crossing property in $(x; t)$ for all $p \in R^n$ if and only if f is supermodular. If $f(\cdot)$ is nondecreasing in x, then $f(x, t) - w \cdot x$ is quasisupermodular in x and has the single crossing property in $(x; t)$ for all nonnegative $w \in R^n$ if and only if f is supermodular.*

Definition 41 $\{h(\cdot; \alpha) : R \to R\}$ *is **richly parameterized** if for all (x', y') and (x'', y'') with $x' \neq x''$, there is some $\hat{\alpha}$ such that $y' = h(x'; \hat{\alpha})$ and $y'' = h(x''; \hat{\alpha})$.*

Theorem 42 *Let $U(x, y, t) : R^3 \to R$ be completely regular with $U_y \neq 0$ and let $h(\cdot; \alpha) : R \to R$ be a richly parameterized family. Then U satisfies the (strict) Spence-Mirrlees condition if and only if for all α, $g(x; t, \alpha) = U(x, h(x; \alpha), t)$ has the (strict) single crossing property in $(x; t)$.*

Remark 43 *This theorem is very useful and usually called the method of dissection: in order to prove the single crossing property of a complex g function, we can transform the part in g function, where no t involves, to U function and prove the Spence-Mirrlees condition of U.*

Corollary 44 *((Milgrom and Shannon 1994): aggregation principle) Let X be a lattice, T a partially ordered set, Y an arbitrary set with $W \subset Y$, $f : X \times Y \times T \to R$, and $x^*(t, S)$ be $\arg\max_{x \in S} \max_{y \in W} f(x, y; t)$. Then $x^*(t, S)$ is monotone nondecreasing in (t, S) if and only if $g(x; t) = \max_{y \in W} f(x, y; t)$ is quasisupermodular in x and satisfies the single crossing property in (x, t).*

Corollary 45 *Let $f : R^2 \to R$ and $h : R \times Y \to R$ and define $x^*(t, S, p) = \arg\max_{x \in S} \max_{y \in W} f(x; t) + h(x, y) + px$. Then $x^*(t, S, p)$ is nondecreasing in t for all $S \subset R$ and all $p \in R$ if and only if f is supermodular. If $f(x; t) + h(x, y)$ is nondecreasing in x for all y, then $x^*(t, S, p)$ is nondecreasing in t for all $S \subset R$ and all $p \leq 0$ if and only if f is supermodular.*

1.3.1 Equilibriums under certainty

Definition 46 ***Supermodular games**: A class of games in which the players' strategy sets S_n are compact sublattices and the payoff functions $\pi_n(x_n, x_{-n}, t)$ are upper semi-continuous in the player's own strategy x_n, continuous in the competitor's strategies x_{-n}, supermodular in (x_n, x_m) where $n \neq m$, and supermodular in (x_n, t) for any fixed values of the other variables.*

Definition 47 ***Game with strategic complementarities****: if for every n :*

1. S_n, player n's strategy set, is compact lattice;

2. π_n, player n's payoff function, is upper semi-continuous in x_n for x_{-n} fixed, and continuous in x_{-n} for fixed x_n;

3. π_n is quasisupermodular in x_n and satisfies the single crossing property in (x_n, x_{-n}).

Theorem 48 *((Milgrom and Shannon 1994) Thm 12) Let Γ be a game with strategic complementarities. Then $\forall n \in N$, there exist strategies x_{*n} and x_n^* which are the smallest and largest serially undominated strategies for player n. Moreover, the pure strategy profiles $x_* = (x_{*n}; n \in N)$ and $x^* = (x_n^*; n \in N)$ are Nash equilibria. (in other words, game with strategic complementarities have at least one pure nash strategy)*

Theorem 49 *Let $\Gamma_t = \{N, (S_n), \pi_n(x_n, x_{-n}, t)\}$ be a family of games with strategic complementarities such that $\pi_n(x_n, x_{-n}, t)$ satisfies the single crossing property in (x_n, x_{-n}, t) for all $n \in N$. Then the largest and smallest pure strategy equilibrium (and serially undominated strategy profiles) $x_{*n}(t)$ and $x_n^*(t)$, are monotone nondecreasing functions of the parameter t.*

1.3.2 Others

Interval Order Dominance

Definition 50 *$f : R^2 \to R$ satisfies* ***interval order dominance*** *iff for any $x^o, x^* \in R$ and any $\theta^o, \theta^* \in R$, if $x^o < x^*$ and $\theta^o \le \theta^*$, then*

$$f(x^*, \theta^o) - f(x, \theta^o) \ge 0 \text{ for } \forall x \in [x^o, x^*] \Rightarrow f(x^*, \theta^*) - f(x^o, \theta^*) \ge 0$$

with a strict inequality if $f(x^*, \theta^o) > f(x, \theta^o)$.

(If f is differentiable, the sufficient condition for interval order dominance is that for any $\theta^o \le \theta^*$, there is a positive, weakly increasing function $\alpha : R \to R$ such that for all $x \in R$, we have $D_x f(x, \theta^*) \ge \alpha(x) D_x f(x, \theta^o)$)

(If α is a constant function, then interval order dominance is equivalent to single crossing.)

Theorem 51 *(Theorem 5 of (Nachbar 2009)) Let $f : R^2 \to R$, let $C \subseteq R$ be an interval, and for each $\theta \in R$, let $\phi(\theta)$ be the set of solutions, assumed non-empty, to the problem*

$$\max_{x \in C} f(x, \theta)$$

If f satisfies the interval order dominance then $\phi(\theta)$ is weakly increasing.

1.4 Comparative statics under uncertainty

Generally, in order to prove $\int g(\cdot) f(\cdot) dt$ to be SC2, we will only need to consider the functional properties of $g(\cdot)$ and $f(\cdot)$, and whether f or g is distribution density or not will be irrelevant.

More specifically, in section "Two functions are log-supermodular functions with choice vector", we only need to separate the objective function $U(X, \theta)$ into $\int g(X, s) f(s, \theta) ds$. And we need g and f to be log-supermodular functions to prove U to be log-supermodular.

In section "One function is single-crossing and the other is log-supmodular with one choice variable", we need to separate the objective function $U(x, \theta)$ into $\int g(x, s) f(s, \theta) ds$ (or even $\int g(x, \theta, s) f(s, \theta) ds$ by using (Athey 2002) Lemma 5 extension 1). Then, we need to shown g is SC2 and f is log-supermodular.

1.4.1 Two functions are log-supermodular functions with choice vector

Agent's objective function is given by $U(X, \theta) = \int u(X, s) f(s, \theta) ds$, where X is choice vector and θ is an exogenous parameter.

$$\text{(MCS)} \quad X^*(\theta, B) = \arg\max_{X \in B} U(X, \theta) \text{ is nondecreasing in } \theta \text{ and } B$$

Lemma 52 *((Athey 2002) Lma 1) Suppose that f is nonnegative. Then (i) MCS holds for all $u : X \times S \to R_+$ log-supermodular, if and only if (ii) U is log-supermodular in (X, θ) for all $u : X \times S \to R_+$ log-supermodular.*

Remark 53 *U is log-supermodular in (X, θ) means U is SC2 in (X, θ) and U is log-supermodular in X. Then utilize (Milgrom and Shannon 1994), it is done for sufficient condition*

This Lemma indicates that if we want MCS to hold for ALL log-supermodular payoffs function u, then objective function U must be log-supermodular for ALL log-supermodular payoffs function u.

Also, if U fails to be log-supermodular for some u log-supermodular, then we can find another log-supermodular payoff v, for which MCS fails.

This lemma does not say anything about the equivalence between MCS and log-sup for some log-sup u.

Lemma 54 *((Ahlswede and Daykn 1978)) Let h_i, $(i = 1, 2, 3, 4)$, represent four nonnegative functions, $h_i : S \to R$. Then*

$$h_1(s) \cdot h_2(s') \leq h_3(s \vee s') \cdot h_4(s \wedge s') \text{ for } \mu\text{-almost for } s,\ s' \in S$$

implies

$$\int h_1(s)\, d\mu(s) \cdot \int h_2(s)\, d\mu(s) \leq \int h_3(s)\, d\mu(s) \cdot \int h_4(s)\, d\mu(s)$$

Lemma 55 *((Athey 2002) Lma 4) Suppose that f is nonnegative, and that $n \geq 2$ if $m \geq 2$ (where m is the dimension of S and n is the dimension of X). Then following two conditions are equivalents: (i) U is log-supermodular in (X, θ) for all $u : X \times S \to R_+$ that are log-supermodular a.e.-μ; (ii) f is log-supermodular in (S, θ) a.e.-μ;*

Theorem 56 *((Athey 2002) Thm 1) Suppose that f is nonnegative, and suppose that $n \geq 2$ if $m \geq 2$ (where m is the dimension of S and n is the dimension of X). The following two conditions are equivalent:*

i) MCS holds for all $u : X \times S \to R_+$ that are log-supermodular a.e.-μ;

ii) f is log-supermodular in (S, θ) a.e.-μ;

Remark 57 *Note: 1. Use the two lemma 1&4 in (Athey 2002) 2. in order to prove U is log-supermodular, it will be sufficient to show u and f is log-supermodular; 3. this Theorem does NOT imply that if MCS hold then u and f is log-supermodular. 4. More generally, for $U(X, \theta) = \int u(X, s, \theta) f(X, s, \theta)\, ds$, as long as u and f are log-supermodular, then U will be log-supermodular and SC2.*

1.4.2 One function is single-crossing and the other is log-supmodular with one choice variable

Agent's objective function is given by $U(x, \theta) = \int u(x, s) f(s, \theta)\, ds$ (or even $\int u(x, \theta, s) f(s, \theta)\, ds$ by using (Athey 2002) Lemma 5 extension 1), where x is choice variable, **only one choice variable**, and θ is an exogenous parameter.

$$\text{(MCS')} \quad x^*(\theta, B) = \arg\max_{x \in B} U(x, \theta) \text{ is nondecreasing in } \theta \text{ and } B$$

Definition 58 *Two hypotheses H-A and H-B are a* ***minimal pair of sufficient conditions (MPSC)*** *for the conclusion C if (i) C holds whenever H-B does, if and only if H-A holds. (ii) C holds whenever H-A does, if and only if H-B holds.*

Theorem 59 *((Athey 2002) Thm 2) (A) u satisfies SC2 in $(x; s)$ a.e.-μ; and (B) f is log-supermodular a.e.-μ; are MPSC for (C) (MSC') holds.*

Remark 60 *Note: again, this theorem does not say that u satisfies SC2 and f is log-supermodular are necessary for (MSC"), but it does indicate that they are sufficient condition. Moreover, if we have u satisfies SC2 (f is log-supermodular) then (MSC') holds if and only if f is log-supermodular (u satisfies SC2).*

(For relaxing condition, please check Lemma 5 and Theorem 3 of (Athey 2002))

Lemma 61 *((Athey 2002) Lma 5: need for the proof of Thm 2) Let* $g : S \to R$ *and* $k : S \times \theta \to R$*. (A)* g *satisfies SC1 a.e.-*μ*; and (B)* k *is log-supermodular a.e.-*μ*; are a MPSC for (C)* $G(\theta) = \int g(s)\, k(s;\theta)\, d\mu(s)$ *satisfies SC1.*

(This Lemma is exactly the same as Theorem 2 by constructing $g(s) = u(x_H, s) - u(x_L, s)$*)*

Lemma 62 *((Athey 2002) Lma 5 extension) Lemma 5 of (Athey 2002) holds under any of the following modifications:*

(i) g *depends on* θ *directly, under the additional restrictions that* g *is piecewise continuous in* θ *and either (a)* g *is nondecreasing in* θ*, or (b) for all* θ*,* g *is nonzero except at single(fixed) point* s_0*, and further, for all* $\theta_H > \theta_L$*,* $g(s, \theta_H)/g(s, \theta_L)$ *is nondecreasing in* s*.*

(ii) We allow that for each θ*, there exists a measure* μ^θ *such that* $K(s;\theta) = \int_{-\infty}^{s} k(t, \theta)\, d\mu^\theta(t)$*, we define* $G(\theta) = \int g(s)\, dK(s;\theta)$*, and we replace (B) with (B'):* θ *orders* $K(\cdot;\theta)$ *by MLR*

(iii) $Supp\,[K(\cdot;\theta)]$ *is constant in* θ*, and (A) is replaced with (A'):* g *satisfies weak SC1.*

Remark 63 *For extension (i): we can extend the* $U(x,\theta) = \int u(x,s)\, f(s,\theta)\, ds$ *to* $U(x,\theta) = \int u(x,\theta,s)\, f(s,\theta)\, ds$

For extension (ii): we can extend the distribution density $f(\cdot)$ *to a probability measure* $F(\cdot)$

For extension (iii): we can reduce the requirement of SC1 to weak SC1.

Theorem 64 *((Athey 2002) Thm 3) For each* $\theta \in \Theta$*, let* $K(\cdot;\theta)$ *be a probability distribution on* S*. Then (C) (MCS') holds for all sets* B *whenever (A)* u *satisfies SC2 in* $(x;s)$ *a.e.-*μ*, and for all* $x_H > x_L$*,* $u(x_H, s) - u(xL, s)$ *is weakly quasiconcave in* s *a.e.-*μ*, if and only if (B)* K *is log-supermodular.*

(Different from Thm 2, this theorem relax the requirement of exist a distribution density function. Instead, it use a distribution function K*. And log-supermodular of* K *is weaker than log-supermodular of* k*.)*

1.4.3 Single crossing of indifference curves

For an arbitrary differentiable function $h : R^3 \to R$ that satisfies $\frac{\partial h(x,y,t)}{\partial y} \neq 0$. Let $V(x,y,\theta) = \int v(x,y,s)\, f(s;\theta)\, d\mu(s)$.

Definition 65 (x,y)*-indifference curves are* ***Well-Behaved (WB)*** *if* h *is differentiable in* (x,y)*;* $\frac{\partial h(x,y,t)}{\partial y} \neq 0$*; the* (x,y)*-indifference curves are closed curves.*

Lemma 66 *((Athey 2002) Lma 8) Let* $v : R^3 \to R$ *and* $f : R^2 \to R_+$*, and suppose that* v *and* V *satisfy (WB). Then (A)* $v(x,y,s)$ *satisfies (SM) a.e.-*μ*, and (B)* f *is log-supermodular in* $(s;\theta)$ *a.e.-*μ*, are a MPSC for (C)* $V(x,y,\theta)$ *satisfies (SM).*

Theorem 67 *((Athey 2002) Thm 4) Lemma 8 of (Athey 2002) also holds if (C) is replaced with* $x^*(\theta, B) = \arg\max_{x\in B} V(x, b(x), \theta)$ *is nondecreasing in* θ *and* B *for all* $b : R \to R$*.*

1.4.4 Summary of (Athey 2002)' results

SUMMARY OF RESULTS

Thm #	A: Hypothesis on u(a.e.-μ)	B: Hypothesis on f(a.e.-μ)	C: Conclusion	Corresponding comparative statics conclusion
Lem 4; Thm 1	$u(\mathbf{x},\mathbf{s}) \geq 0$ is log-spm.	$f(\mathbf{s};\theta)$ is log-spm.	$\int u(\mathbf{x},\mathbf{s})f(\mathbf{s};\theta)\, d\mu(\mathbf{s})$ is log-spm in $(\mathbf{x},\theta)$.	$\arg\max_{x\in B} \int u(\mathbf{x},\mathbf{s})f(\mathbf{s};\theta)\, d\mu(s)$ $\uparrow$ in θ and B.
Lem 5; Thm 2	$u(x,s)$ satisfies SC2 in $(x;s)$.	$f(s;\theta)$ is log-spm.	$\int u(x,s)f(s;\theta)\, d\mu(s)$ satisfies SC2 in $(x;\theta)$.	$\arg\max_{x\in B} \int u(x,s)f(s;\theta)\, d\mu(s)$ $\uparrow$ in θ and B.
Lem 7; Thm 3	$u(x,s)$ satisfies SC2 and the returns to x are quasi-concave in s.	$F(s;\theta) \geq 0$ is log-spm.	$\int u(x,s)f(s;\theta)\, d\mu(s)$ satisfies SC2 in $(x;\theta)$.	$\arg\max_{x\in B} \int u(x,s)f(s;\theta)\, d\mu(s)$ $\uparrow$ in θ and B.
Lem 8, Thm 4	$u(x,y,s)$ satisfies SM.	$f(s;\theta)$ is log-spm.	$\int u(x,y,s)f(s;\theta)\, d\mu(s)$ satisfies SM.	$\arg\max_{x\in B} \int u(x,b(x),s)f(s;\theta)\, d\mu(s)$ $\uparrow$ in θ and B for all $b : \mathbb{R} \to \mathbb{R}$.

In rows 1, 2, and 4: (A) and (B) are a *minimal pair of sufficient conditions* (Definition 4) for the conclusion (C); further, (C) is equivalent to the comparative statics result in column 4. In row 3, the same relationships hold except that (A) is not necessary for (C) to hold whenever (B) does.

Notation and Definitions. Bold variables are real vectors; italicized variables are real numbers; f is nonnegative; log-spm indicates log-supermodular (Definition 3); sets are increasing in the strong set order (Definition 1); SC2 indicates single crossing of incremental returns to x (Definition 2); and SM indicates single crossing of $x - y$ indifference curves (Section V). Arrows indicate weak monotonicity.

Note that those conditions are "necessary and sufficient" under the terminology of minimal pair of sufficient conditions (MPSC). (Not the traditional Necessary and Sufficient condition.)

Also, there is an extension for "One function is single-crossing and the other is log-supmodular with one choice variable" section ((Athey 2002) Thm 2), we can separate the objective function $U(x,\theta)$ into $\int u(x,\theta,s)\, f(s,\theta)\, ds$ by using (Athey 2002) Lemma 5 extension 1.

1.5 Aggregating single crossing property

In (Athey 2002), we know the condition that the single crossing property holds for single integral. But the multi-dimensional integrals case were not discussed in details. In (Quah and Strulovici 2010), authors want to find the condition single crossing property preserved under single or multiple integrals. Also, authors generalize the relax some conditions in (Athey 2002).

In this section, according to (Quah and Strulovici 2010), **we assume the action space / decision space is one dimension**.

1.5.1 Single integral

In this section, we want to find the condition on $f : \Theta_1 \times \Theta_2 \to R$ which guarantee that $F : \Theta \to R$ to be SC1 ($\mathcal{S}$ functions), where

$$F(\theta_1) = \int_{\Theta_2} f(\theta_1, \theta_2)\, d\theta_2$$

In order to prove $\int_{\Theta_2} f(\cdot)\, g(\cdot)\, d\theta_2$ be SC1, instead of requiring f to be SC1 and g be logsupermodular function as in (Athey 2002), (Quah and Strulovici 2010) relax this the condition by only requiring $f \cdot g$ be $\mathcal{S}$-summable family. (By the proof of Corollary 1: if f is SC1 and g is logsupermodular, then $f \cdot g$ is $\mathcal{S}$-summable family)

Definition 68 *((Quah and Strulovici 2010)) Define the* ***binary relation*** $\sim$ *by the following way: we say that* $h \sim g$ *if*

a) at any $\theta' \in \Theta$, *such that* $g(\theta') < 0$ *and* $h(\theta') > 0$, *we have*

$$-\frac{g(\theta')}{h(\theta')} \geq \frac{g(\theta'')}{h(\theta'')} \text{ when } \theta'' > \theta'; \text{ and}$$

b) at any $\theta' \in \Theta$, *such that* $h(\theta') < 0$ *and* $g(\theta') > 0$, *we have*

$$-\frac{h(\theta')}{g(\theta')} \geq \frac{h(\theta'')}{g(\theta'')} \text{ when } \theta'' > \theta'.$$

Remark 69 $\sim$ *is a reflexive relation and is not transitive.*

If $h \sim g$ *then* $\alpha h \sim \beta g$ *for* α *and* β *are nonnegative scalars.*

Definition 70 *((Quah and Strulovici 2010)) A function with SC1 is called* $\mathcal{S}$ ***function***

Definition 71 *((Quah and Strulovici 2010)) Two SC1 functions that are related by* $\sim$ *is called* $\mathcal{S}-$***summable****; a family of SC1 functions in which any two functions are related to each other is said to be an* $\mathcal{S}-$***summable family****.*

Proposition 72 *((Quah and Strulovici 2010) Prop 1) Let* h *and* g *be two* $\mathcal{S}$ *functions. Then* $\alpha h + g$ *is an* $\mathcal{S}$ *function for all positive scalars* α *if and only if* $h \sim g$.

Proposition 73 *((Quah and Strulovici 2010) Prop 2) Suppose* $\{f_i\}_{1 \leq i \leq M}$ *is an* $\mathcal{S}-$*summable family. (i) Then* $\sum_{i=1}^{M} \alpha_i f_i$ *where* $\alpha_i \geq 0$ *for all* i, *is an* $\mathcal{S}$ *function. (ii) suppose* h *is an* $\mathcal{S}$ *function and* $h \sim f_i$ *for all* i, *then* $h \sim \sum_{i=1}^{M} f_i$.

Theorem 74 *((Quah and Strulovici 2010) Thm 2) Let* T *be a measurable subset of* $\mathbb{R}$ *and* $\{f(\cdot, \theta_2)\}_{\theta_2 \in \Theta_2}$ *an* $\mathcal{S}$*-summable family indexed by elements in* Θ_2 *and defined on* Θ_1 $\left(f(\cdot, \theta_2') \sim f(\cdot, \theta_2'')\right.$ *for any* $\left.\theta_2', \theta_2'' \in \Theta_2\right)$. *For any fixed* θ_1, $f(\theta_1, \cdot)$ *is a measurable and bounded function of* θ_2. *(i) Then the function* $F : \Theta_1 \to \mathbb{R}$ *defined by* $F(\theta_1) = \int_{\Theta_2} f(\theta_1, \theta_2)\, d\theta_2$ *is also an* $\mathcal{S}$ *function. (ii) If* g *is an* $\mathcal{S}$ *function and* $g \sim f(\cdot, \theta_2)$ *for all* $\theta_2 \in \Theta_2$, *then* $g \sim F$.

Corollary 75 *((Quah and Strulovici 2010)) ((Athey 2002) Lemma 5) Let* T *be a measurable subset of* $\mathbb{R}$ *and* K *a subset of* $\mathbb{R}$. *Suppose* $f : T \to \mathbb{R}$ *is an* $\mathcal{S}$ *function and that* $g : T \times K \to \mathbb{R}_{++}$ *is a logsupermodular function. Then* $F : K \to \mathbb{R}$ *is an* $\mathcal{S}$ *function, where* $F(k) = \int_T f(t) g(k, t)\, dt$.

Remark 76 *(Quah and Strulovici 2010) generalized the results of (Athey 2002) Lemma 5: if f is SC1 and g is logsupermodular, then $f \cdot g$ is $\mathcal{S}$-summable family. Hence, (Athey 2002)'s results can be further generalized into the $\mathcal{S}$-summable family.*

Proposition 77 *Suppose $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$ and let f and g be a bounded and measurable function defined on this interval. For some point $\hat{\theta}$ in the interior of Θ, and $a \leq \hat{s}$, define the function $\hat{f} : \{a\} \cup (\hat{\theta}, \overline{\theta}] \to \mathbb{R}$ by*

$$\hat{f}(\theta) = \begin{cases} \int_{[\underline{\theta},\hat{\theta}]} f(z)\, dz & \text{if } \theta = a \\ f(\theta) & \text{if } \theta \in (\hat{\theta}, \overline{\theta}] \end{cases}$$

This process is called domain coarsening (Domain coarsening preserves the single crossing property.) Define $\bar{f}$ on the two-point domain $\{0, 1\}$ by

$$\bar{f}(\theta) = \begin{cases} \int_{[\underline{\theta},\hat{\theta}]} f(z)\, dz & \text{if } \theta = 0 \\ \int_{(\hat{\theta},\overline{\theta}]} f(z)\, dz & \text{if } \theta = 1 \end{cases}$$

Then (i) The function $\hat{f}$ and $\hat{g}$ satisfy $\hat{f} \sim \hat{g}$. (ii) the function $\bar{f}$ and $\bar{g}$ satisfy $\bar{f} \sim \bar{g}$

1.5.2 Multiple integral

In this section, we consider functions defined on the domain $\Theta = \prod_{i=1\Theta i}^{n}$, with Θ_i a bounded and measurable subset of $\mathbb{R}$. For any $\theta \in \Theta$, we denote its subvector consisting of entries in $K \subset N = \{1, 2, ..., n\}$ by θ_K and write θ as $\left(\theta_{N\setminus K}, \theta_K\right)$. The set consisting of the subvectors θ_K we denote by Θ_K, so $\Theta = \Theta_{N\setminus K} \times \Theta_K$. If we restrict f function to the subvector $\Theta_{N\setminus K}$ with a fixed value θ'_K, we denote this restricted function as $f\left(\cdot, \theta'_K\right)$. Denote N_k as a subset of N that $N_k = N\setminus\{k\}$.

In this section, we want to find conditions on $f : \Theta \to \mathbb{R}$ which guarantee that $F : \Theta_1 \to \mathbb{R}$ is SC1, where

$$F(\theta_1) = \int_{\Theta_2} \int_{\Theta_3} \cdots \int_{\Theta_n} f(\theta_1, \theta_2, ..., \theta_{n-1}, \theta_n)\, d\theta_n d\theta_{n-1} ... d\theta_2$$

Definition 78 *A function $f : \Theta \to \mathbb{R}$ has the j-**integrable single crossing property** if it is SC1 ($\mathcal{S}$ function) on $\Theta = \prod_{i=1\Theta i}^{n}$ and*

$$f\left(\cdot, \theta''_K\right) \sim f\left(\cdot, \theta'_K\right)$$

*whenever $\theta''_K > \theta'_K$, for every $K \subset N_j$. We refer to such a function as an $\mathbb{I}_j$ **function**.*

Remark 79 *Since f is SC1, if $f\left(\theta^*_{N\setminus K}, \theta''_K\right)$ and $f\left(\theta^*_{N\setminus K}, \theta'_K\right)$ have opposite signs then it must be the case that the former is positive and the latter is negative, hence $f\left(\cdot, \theta''_K\right) \sim f\left(\cdot, \theta'_K\right)$ condition is equivalent to checking $-\frac{f\left(\theta^*_{N\setminus K}, \theta'_K\right)}{f\left(\theta^*_{N\setminus K}, \theta''_K\right)} \geq -\frac{f\left(\theta^{**}_{N\setminus K}, \theta'_K\right)}{f\left(\theta^{**}_{N\setminus K}, \theta''_K\right)}$ whenever $\theta^{**}_{N\setminus K} > \theta^*_{N\setminus K}$.*

Definition 80 *If f is an $\mathbb{I}_j$ function for every $j \in N$, then we shall refer to it as an $\mathbb{I}$ **function***

Remark 81 *If f is an $\mathbb{I}_j$ function and g is logsupermodular, then $h(\theta) = f(\theta) g(\theta)$ is an $\mathbb{I}_j$ function.*

Any increasing function is an $\mathbb{I}$ function.

Proposition 82 *((Quah and Strulovici 2010) Prop 6) Let $f : \Theta \to \mathbb{R}$ be an $\mathcal{S}$ function. Then f is an $\mathbb{I}_1$ function if and only if the following holds: $f\left(\cdot, \theta''_K\right) \sim f\left(\cdot, \theta'_K\right)$ whenever $\theta''_K > \theta'_K$, where K is a subset of N with exactly $n-1$ elements and $\theta'_1 = \theta''_1$ if $1 \in K$.*

Theorem 83 *((Quah and Strulovici 2010) Thm 3) Let $f : \Theta \to \mathbb{R}$ be a bounded and measurable $\mathbb{I}_1$ function. Then (i) $F_n : \Theta_{N_n} \to \mathbb{R}$ as defined by $F_n\left(\theta_{N_n}\right) = \int_{\Theta_n} f\left(\theta_{N_n}, \theta_n\right) d\theta_n$ is an $\mathbb{I}_1$ and (ii) $F : \Theta_1 \to \mathbb{R}$ as defined by $F(\theta_1) = \int_{\Theta_2} \int_{\Theta_3} \cdots \int_{\Theta_n} f(\theta_1, \theta_2, ..., \theta_{n-1}, \theta_n)\, d\theta_n d\theta_{n-1} ... d\theta_2$ is an $\mathcal{S}$ function.*

Corollary 84 *((Quah and Strulovici 2010) Corollary 2) (Karlin and Rinott 1980) ((Ahlswede and Daykn 1978)) Let* X_i *(for* $i = 1, 2, ..., m$*) and* Y_j *(for* $j = 1, 2, ..., m$*) be measurable subsets of* $\mathbb{R}$ *and suppose that the function* $\phi : X \times Y \to R$ *(where* $X = \prod_{i=1}^{m} X_i$ *and* $Y = \prod_{j=1}^{n} Y_j$*) is uniformly bounded and measurable with respect to* $y \in Y$*. If* ϕ *is logsupermodular in* (x, y)*, then the function* $\Phi(x) = \int_Y \phi(x, y)\, dy$ *is also a logsupermodular function.*

Proposition 85 *((Quah and Strulovici 2010) Prop 7) Let* $f : \Theta \to \mathbb{R}$ *be bounded and measurable* $\mathbb{I}_1$ *function and suppose that* $g \sim f\left(\cdot, \theta'_{N_1}\right)$ *for every* $\theta'_{N_1} \in \Theta_{N_1}$*, where* $g : \Theta_1 \to \mathbb{R}$ *is an* $\mathcal{S}$ *function. Then* $g \sim F$*, where* F *is defined by* $F(\theta_1) = \int_{\Theta_2} \int_{\Theta_3} \cdots \int_{\Theta_n} f(\theta_1, \theta_2, ..., \theta_{n-1}, \theta_n)\, d\theta_n d\theta_{n-1} ... d\theta_2$*.*

Theorem 86 *((Quah and Strulovici 2010) Thm 4) Let* $V(x; \theta_1) = \int_{\Theta_{N_1}} v(x; \theta_1, \theta_2, ..., \theta_n)\, f(\theta_{N_1} | \theta_1)\, d\theta_{N_1}$ *and* $\Delta(\theta) = v(x''; \theta) - v(x; \theta)$*. Suppose* $f(\theta_{N_1} | \theta_1)$ *is logsupermodular and, for any* $x'' > x'$*,* Δ *is an* $\mathbb{I}_1$ *function. Then* $V(x; \theta_1)$ *obeys single crossing differences and* $\arg\max_{x \in X} V(x; \theta_1)$ *increases with* θ_1*.*

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