The Role of Product and Market Information in an Online Marketplace

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Although the online marketplaces have flourished, the role of product and market information in how it affects an online marketplace and how it can be leveraged to improve the sales or sales volume of the platform has not been explored in the prior research. We study how the provision of product and/or market information affects buyers' and sellers' behavior in an online marketplace by establishing the (Pareto-dominant) equilibrium for the sellers' pricing decisions under various information structures. Our main findings are as followings. First, we show that in equilibrium while sales volume of the platform increases in both the size of the buyers' pool and the size of the sellers' pool, sales increase only in the size of the buyers' pool and are unimodal in the size of the sellers' pool. Second, by analytically characterizing the platform's optimal information strategy as a function of the underlying market parameters and whether the platform's goal is to maximize sales or sales volume, we find that providing product and/or market information may backfire on the platform by jeopardizing its financial performance. Third, we demonstrate using numerical studies that information is more valuable to the platform when the goal of the platform is to maximize sales rather than sales volume, and when it faces a seller's market (i.e., demand-to-supply ratio is greater than one) rather than a buyer's market (i.e., demand-to-supply ratio is less than one).

Key words: online marketplace; information strategy; equilibrium analysis; value of information.

1. Introduction

The rise of the Internet has created a spurt for online marketplaces, with 97% of U.S. consumers who shop online do so in a marketplace as of 2017 (Ali 2018). An online marketplace is a website or a mobile application where potential buyers have access to buy products offered by many different sellers. The operator of a marketplace typically operates on a consignment basis, meaning that it does not own the inventory. Rather, the operator provides a platform for sellers and buyers to list and browse products, respectively, and for the two sides to execute secure transactions. Leading online marketplaces include Alibaba (Taobao), Amazon Marketplace, eBay, and JD.com, but there are many others such as Buy.com, Etsy, Newegg, Rakuten, Shopify, Swappa, just to name a few. To put things in perspective, half of all online sales in 2016 occurred in online marketplaces (Howland 2017), with this number only expected to grow moving toward 2020.

Typically, an online marketplace makes money by either charging customers a fixed commission (i.e., fee per transaction) or a proportional commission (i.e., fee as a percentage of sales). When a fixed commission is charged by the online marketplace (e.g., Swappa), the platform's revenue increases proportionally with the increase in total sales volume (i.e., total number of transactions). In comparison, if a proportional commission is charged (e.g., eBay), the platform's revenue increases proportionally with total sales (i.e., total dollar amount of all transactions). As a result, sales and sales volume are the standard metrics to measure the platform's financial success.

To attract sellers and buyers to visit its platform and complete transactions, an online marketplace can invest in platform resources and technology. The online marketplace can also leverage different information structures to increase sales and/or sales volume, which is the core focus of our study. In particular, we distinguish two types of information that can be provided by an online marketplace on its platform, namely, *product information* and *market information*.

Product information, in this paper, refers to *detailed* description of a product in addition to any basic textual information. It can include technical specifications and product demonstrations shown in graphics and videos as well as Q&A's and customer reviews that will allow customers to better understand and evaluate the product according to their individual needs. Without such information, the buyers may not be able to accurately value the product which ultimately will affect their willingness-to-pay for the product. Although some online marketplaces allow individual sellers to post the product information on their own, we focus in this paper on the ones where it is the platform that determines the amount and type of product information to be released. Examples of such online marketplaces include Amazon, Submarino (leading online marketplace in Brazil), Real.de (top online marketplace in Germany) and many others. Interestingly, while Amazon provides product information (videos, Q&A's, and customer reviews) on its website, Submarino and Real.de do not[1], illustrating that these platforms have the ability to control product information.

Market information, on the other hand, refers in this paper to the actual demand and supply of a product in the online marketplace. This information can be obtained by the platform through real-time data collection (i.e., tracking website visitors and product listings) and analysis (i.e., deploying machine learning algorithms). For example, Google Analytics or equivalent in-house data tools can inform a platform how many visitors are on a particular product page in real-time (Lovestrand

[1] For example, see the respective listing for Bose QuietComfort 35 Series II Wireless Headphones at Amazon (`https://www.amazon.com/Bose-QuietComfort-Wireless-Headphones-Cancelling/dp/B0756CYWWD/ref=sr_1_3?`), Submarino (`https://www.submarino.com.br/produto/37973955/fone-de-ouvido-bose-quietcomfort-35-ii-wls-789564-0010-preto?`), and Real.de (`https://www.real.de/product/326945939/?`).

2019). Note that the market information is not readily available to sellers and buyers unless the platform, who owns the website of the online marketplace, chooses to release it. Releasing the market information or not, is however a delicate decision of the platform because access to market information could potentially change the sellers' pricing decisions for a product as it enables them to better estimate the relationship between the product's selling probability and price.

Despite the unprecedented boom in the online marketplaces and the proliferation of information in contemporary life, there is not much research on the role of product and market information in an online marketplace and the resulting decisions of buyers and sellers involved in the existing literature. To this extent, we take an important first step in this paper to understand the following research questions:

Q1. How do sellers make their optimal pricing decisions under different product and market information structures in an online marketplace?

Q2. What is the impact of information on sales and sales volume of the platform?

Q3. When should an online marketplace provide the buyers and sellers with product and/or market information on its platform?

To address these questions, we develop a stylized model to capture four different information structures of an online marketplace: (i) In the no-information case, the platform does not provide its buyers and sellers with the product or market information; (ii) In the product-information case, only the product information is provided to the buyers and sellers; (iii) In the market-information case, only the market information is provided; and finally, (iv) In the product-and-market-information case, both the product and market information is provided.

In the model, the number of buyers and sellers who join the platform are randomly determined from a buyers' pool and a sellers' pool, respectively. The sellers make pricing decisions for their products by considering not only buyers' product valuations but also other sellers' strategies, necessitating an equilibrium analysis. While the market information directly changes sellers' pricing decisions in equilibrium, the product information directly changes buyers' product valuations and, therefore, indirectly affects the equilibrium prices and financial performance of the platform.

We report the following main results in this paper. First, the (Pareto-dominant) equilibrium is established for the sellers' pricing decisions under all information structures being investigated (no information, product information, market information, product-and-market information). We find that sales volume of the platform increases with the size of either the buyers' pool or the sellers' pool, however, sales of the platform only increase with the size of the buyers' pool but exhibit a non-monotonic pattern with an increasing size of the sellers' pool.

Second, by comparing the sales and the sales volume between the different information structures, we show that providing product and/or market information can actually be harmful to the platform.

We further identify the optimal information strategy for the platform, which is governed by the underlying market parameters. In particular, we find that it is optimal for the platform to provide both the product and market information in a seller's market and not to provide the product information in a buyer's market, regardless of whether the platform's goal is to maximize sales or sales volume. In addition, in a buyers' market, providing the market information benefits the platform only when its goal is to maximize sales, but not to maximize sales volume.

Finally, we quantify the value of information through numerical experiments. We observe that under the same market parameters, information is more valuable to the platform when the goal of the platform is to maximize sales rather than sales volume, and when it faces a seller's market rather than a buyer's market.

The remainder of the paper is organized as follows. §2 reviews the related literature. §3 introduces the model framework. §4 sets up the no-information case as a benchmark and formalizes the performance metrics of the platform. §5 studies three informational cases. §6 derives the optimal information strategy for the platform. §7 illustrates the value of information. And finally, §8 concludes the paper. Proofs are relegated to the Electronic Companion of the paper.

2. Literature Review

Our research is closely related to the literature on information control strategy, and we study two types of information in this paper, namely product information and market information.

Provision of the product information can influence buyers' product valuations. Lewis and Sappington (1991, 1994) extend the principal-agent model to study the incentives of a seller to release product information to potential buyers, concluding that the optimal product information decision for the seller is an all-or-nothing strategy: either release all the product information or does not release at all. Johnson and Myatt (2006) further justify the all-or-nothing product information revelation strategy established in Lewis and Sappington (1991, 1994) by a U-shaped function of dispersion for buyers' valuations. In contrast to static models in which the seller and buyers interact only once (as in Lewis and Sappington 1991, 1994 and Johnson and Myatt 2006), Chu and Zhang (2011) develop a dynamic two-period model to study the joint impact of the optimal product information strategy and the optimal pricing strategy under a preorder setting, finding that the seller should release some product information or none, but should never release all information. In this paper, we consider an online marketplace's optimal product (and market) information revelation strategy. Hence, different from the above-mentioned papers, the information in our case is possessed by a two-sided market operator as opposed to a seller.

On the other side, provision of the market information can directly influence sellers' decisions. Research related to the market information in the Operations Management literature typically targets the demand-side market information (i.e., information regarding the number of end customers). Information sharing decisions are typically studied under vertical supply chain settings, where the downstream firm has the demand information and could potentially release it to the upstream firm. There is an extensive literature on assessing the value of demand information (see, e.g., Chen 1998, Gavirneni et al. 1999, Cachon and Fisher 2000, Lee et al. 2000, Zhao and Simchi-Levi 2002, Li and Zhang 2013), and various mechanisms have been proposed to encourage downstream firms to truthfully reveal and share demand information (e.g., Cachon and Lariviere 2001, Özer and Wei 2006). Different from this stream of literature, the platform considered in our paper possesses both demand-side and supply-side market information and may decide to release the market information to influence the equilibrium listing price distribution in the marketplace.

Our research is also broadly related to the literature of online marketplaces. A marketplace can provide information to participating buyers and sellers to facilitate the exchange of products, creating economic value for buyers, sellers, and society at large (Bakos 1991, 1998). Its role and influential factors (such as market size, prices, commission model, product design, product fulfillment, etc.) have been an active research topic in the Economics, Operations, and Information System literatures (see, e.g., Parker and Van Alstyne 2005, Hagiu 2009, Chen et al. 2016, Lai et al. 2018, Li and Netessine 2018, Xiao and Xu 2018). Recently, there is an increasing interest in studying the influence of a marketplace under various supply chain strategies (see Grieger 2003, Wang et al. 2008 for detailed reviews). For example, Ryan et al. (2012) consider a single retailer contemplating selling through a new marketplace channel, in addition to a traditional direct channel, to generate additional demand. The retailer either pays a fixed or proportional commission to the platform in the dual-channel setting, and the authors characterize the optimal pricing decisions for both the retailer and the platform. Jiang et al. (2017) consider a retailer who orders products from a manufacturer and sells to customers through an online marketplace platform where the customers can then resell products later on. In this case, the authors examine the impact of the fixed commission charged by the platform for each transaction on consumer surplus and social welfare. Different from these works, our paper focuses on the platform's information strategies.

Another stream of the literature explores the determination of prices in the marketplaces (e.g., Campbell et al. 2005, Kuruzovich et al. 2010) and the design of efficient trading mechanisms to match buyers and sellers (e.g., Wilson 1985, Chu and Shen 2006). In particular, McAfee (1992) examines the efficiency of the double auction mechanism among buyers and sellers, and Su (2010) proposes a matching mechanism in an online marketplace, which can be interpreted as a posted-price mechanism with efficient rationing and retains similar matching outcome as the double auction. The matching mechanism adopted in our paper directly follows Su (2010).

3. Model Framework

We consider an online marketplace (hereafter referred to as the"platform"). The platform orchestrates a two-sided market where a number of sellers try to the same new product to a number of buyers. Each seller has one unit of the product in stock, and each buyer is interested in one unit of the product. Sellers and buyers make fully rational pricing and purchasing decisions, respectively, in order to maximize their individual utilities. The platform on the other hand can influence the decisions of both sides by providing (or not providing) product and/or market information. Details on the information structure will be given shortly.

The number of sellers to join the platform is described by a random variable D_s, which follows a uniform distribution between 0 and s, i.e., $D_s \sim U[0,s]$. We will refer to s as the *size of the sellers' pool* hereinafter. Each seller sets a price p for the product she owns in order to maximize her individual expected revenue, subject to a reservation value constraint. The reservation value, denoted by v_s, is idiosyncratically developed and represents the minimum price that a seller is willing to go with for listing her product. Even though v_s represents a deterministic value for each seller, a distribution is formed among all potential sellers. We will normalize it to a standard uniform distribution $U[0,1]$ and denote the corresponding cumulative distribution function (CDF) for v_s as $F_s(\cdot)$.

Formally, consider an arbitrary seller with the reservation value v_s. The goal of this seller is to determine the listing price for the product, in order to maximize her individual expected revenue, subject to the reservation value constraint, i.e.,

$$\max_p \{p \cdot P(p)\} \text{ subject to } p \geq v_s. \tag{1}$$

Here in (1), $P(p)$ denotes the probability of the seller successfully selling the product at price p and it will be determined by the market force. In particular, $P(p)$ depends on the pricing decisions of all the sellers and the willingness-to-pay of all the buyers who join the platform. We will specify $P(p)$ in detail later in the paper when the equilibrium concept is introduced. For convenient reference, the main notations used throughout the paper are summarized in Table 1.

The number of buyers that will join the platform is determined by a random variable $D_b \sim U[0,b]$, where b will be referred to as the *size of the buyers' pool*. Buyers are heterogeneous in their valuations v_b for the product. As there are multiple buyers and sellers in the marketplace, we follow the matching mechanism developed in Su (2010) to determine the outcome of the transactions: the product offered with the lowest price p (with ties broken randomly) will be acquired by the buyer with the highest valuation v_b (with ties broken randomly) as long as $p \leq v_b$, and then the second lowest-priced unit will be acquired by the buyer who has the second-highest valuation, and so on

Table 1 Notations

Symbol	Description
$D_s \sim U[0,s]$	random number of sellers where s is the size of the sellers' pool
p	product listing price
v_s	sellers' reservation value with CDF $F_s(\cdot)$
$P(p)$	probability of successfully selling the product at price p
$\tilde{F}_s(\cdot), \tilde{f}_s(\cdot)$	CDF and PDF of the sellers' equilibrium listing price distribution
$I_s \in \{0,1\}$	whether market information (for the sellers) is provided by the platform
$D_b \sim U[0,b]$	random number of buyers where b is the size of the buyers' pool
v_b	buyers' valuation for the product with CDF $F_b(\cdot)$
$p_{max}(d_b, d_s)$	highest or maximum transaction price given market size realizations d_b, d_s
$I_b \in \{0,1\}$	whether product information (for the buyers) is provided by the platform
T	expected sales
Q	expected sales volume
Model N	no-information model, i.e., $I_b = 0$ and $I_s = 0$
Model P	product-information(-only) model, i.e., $I_b = 1$ and $I_s = 0$
Model M	market-information(-only) model, i.e., $I_b = 0$ and $I_s = 1$
Model B	(both-)product-and-market-information model, i.e., $I_b = 1$ and $I_s = 1$

and so forth, until there are no more available products, or no more available buyers, or the prices of the remaining products are all higher than product valuations by the remaining buyers. Note that this matching mechanism can be achieved by either a double auction (Wilson 1985, McAfee 1992) or efficient rationing (Su 2010), and the rational of this mechanism lies in the principle that the higher-valuation buyers are more self-motivated to purchase the product, so they are more likely to insert efforts to search for the product and/or make an earlier purchase, which typically leads to a better price deal.

Next, we describe the platform which can provide or withhold two different types of information on its website, namely, the (detailed) product information and the market information.

Product information refers to detailed product description and technical specifications, Q&A's about the product, and consumer reviews, all of which can facilitate the decision-making process of the buyers. We use the indicator parameter $I_b \in \{1,0\}$ to denote whether the product information is provided by the platform. When buyers are better informed of the product, they will form more accurate individual valuations of the product, i.e., the provision of product information to the buyers leads to a larger spread of product valuations among them (Chu and Zhang 2011). For example, consider two buyers interested in buying a printer where buyer one is looking for the borderless printing feature and buyer two for the Ethernet networking. When the detailed product information is provided, say the printer does support borderless printing but does not support Ethernet networking, then the printer is deemed a higher valuation to buyer one than buyer two. Therefore, when $I_b = 1$, through the detailed product information provided, the buyers can deduce their exact valuations of the product, and we set $v_b \sim U[0,1]$ to model the buyers' differing valuation preferences of the product. In contrast, if the detailed product information is not provided by the platform, i.e., $I_b = 0$, the buyers cannot accurately assess whether the product will fulfill their individual needs. In our example, it implies that buyer one and buyer two do not know if the printer supports borderless printing or Ethernet networking. As a result, we assume for simplicity that all the buyers will evaluate the product at the expected value of all possible valuations, i.e., $v_b \equiv \mathbb{E}(U[0,1]) = 0.5$ for all buyers. We denote the CDF of buyers' product valuation distribution, regardless of its exact distribution (whether $v_b \sim U[0,1]$ or $v_b \equiv 0.5$), as $F_b(\cdot)$.

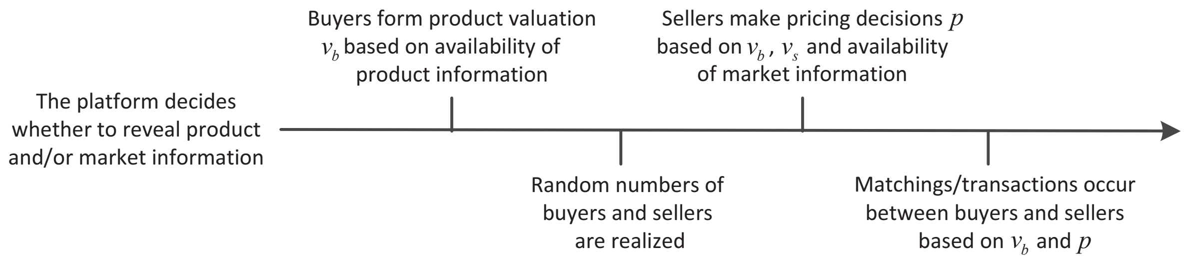
Thanks to the development of data collection methods (including the visitor tracking technology like Google Analytics or equivalent in-house data tools), the platform is able to observe how many sellers and buyers are in the market, i.e., the realized market sizes of D_s and D_b, which will be denoted by d_s and d_b respectively and referred to collectively as the market information. Provision of the market information will have an impact on the sellers' pricing decisions, because the sellers are optimizing their strategies by taking into account the selling probability function $P(\cdot)$, see (1), and $P(\cdot)$ depends on the platform's information structure. We will use the indicator parameter $I_s \in \{1, 0\}$ to denote whether the platform reveals the market information to the sellers (and buyers). In particular, if $I_s = 1$, the sellers possess the information of the values of d_s and d_b when calculating $P(\cdot)$ and pricing their products accordingly. By contrast, if $I_s = 0$, then what the sellers know is only the distributional information, i.e., D_s (resp. D_b) is uniformly distributed between 0 and s (resp. between 0 and b).

Note that the market information affects the $\underline{\text{s}}$ellers' decisions but not the buyers' product valuations, so we choose to use I_s to denote the corresponding revelation decision of the market information by the platform. In comparison, the product information described earlier affects the $\underline{\text{b}}$uyers by forming their product valuation distribution (which then indirectly affects the sellers' decisions), so we use I_b to denote whether the product information is provided by the platform. Furthermore, we have assumed uniform distributions to glean analytical results, however, our main results are preserved under more general distributions such as the Beta distribution.

While each seller acts to maximize their expected revenue, so does the platform in our model. In practice, marketplaces make money by charging customers for commissions/transaction fees. Typically, the commission either increases proportionally with the transaction value (i.e., a percentage of sales) or is fixed (i.e., a flat fee per each transaction). Hence, we study the profitability of the platform in our model by considering two performance metrics, namely sales (i.e., total dollar amount of all transactions) and sales volume (i.e., total number of transactions). The optimal commission problem is not considered in this paper, and it is clear that for any *given* commission percentage (resp. flat fee), the platform's revenue increases in the sales (resp. sales volume).

Finally, we summarize the sequence of events for our model which is captured in Figure 1. First, the online marketplace determines whether it should provide product information and/or

Figure 1 Sequence of Events



market information on its platform. Based on the availability of the product information, buyers form their valuations for the product (v_b). Then, random numbers of sellers and buyers (D_s and D_b) are realized. Next, the sellers make pricing decisions based on v_b and the availability of the market information, subject to their individual reservation value (v_s). In particular, all sellers see if product information is being offered by the platform to the buyers, so that the sellers can infer whether $v_b \sim U[0,1]$ or $v_b \equiv 0.5$. Each seller makes a rational and optimal pricing decision by taking into account the other sellers' strategies. We model such competitive interaction among the sellers using the *Nash equilibrium* concept and refer to it as the *sellers' game*. The CDF of the sellers' listing price distribution in equilibrium is of our interest and will be denoted as $\tilde{F}_s(\cdot)$. Lastly, matchings/sales occur between the buyers and sellers based on the mechanism described earlier.

In what follows, we present four models to study the role of product and/or market information. Specifically, §4 focuses on the no-information model as a benchmark, i.e., $I_b = 0$ and $I_s = 0$, and §5 examines the other three (informational) models where $I_b = 1$ or/and $I_s = 1$.

4. No-Information Model

We first consider the no-information model, or Model N, i.e., when $I_b = 0$ and $I_s = 0$. Recall that when the platform does not provide the product information, the buyers cannot accurately assess their personal utilities of the product. Hence, they evaluate the product at its expected value, i.e., $v_b \equiv \mathbb{E}(U[0,1]) = 0.5$. On the other hand, without the market information, sellers only know the distributional information of D_b and D_s, but not the realized market size d_b and d_s. The goal of this section is to solve the sellers' game, identify the matchings/transactions between the buyers and the sellers, and evaluate the corresponding performance metrics under the no-information model. First, we proceed to characterize the sellers' pricing decisions in equilibrium.

Let $\tilde{F}_s^N(\cdot)$ denote the CDF of the equilibrium listing price distribution of the sellers in Model N (we use the superscript N to denote quantities under the no-information setting). Note that the probability for any two sellers to have the same reservation value v_s is zero, but sellers with different reservation values can impose the same listing price in equilibrium. The selling probability function can be expressed as follows.

$$P^N(p) = \begin{cases} \underset{D_s, D_b}{\mathbb{E}} \left[\mathbb{1}\left\{ D_b \geq D_s \cdot \tilde{F}_s^N(p) \right\} \right] & \text{if } p \leq 0.5; \\ 0 & \text{if } p > 0.5, \end{cases} \tag{2}$$

where $\mathbb{1}\{\cdot\}$ is the standard indication function. Because $v_b \equiv 0.5$, no buyer is willing to acquire the product for a price higher than 0.5, thus the selling probability is zero when $p > 0.5$, which is reflected by the second line of Eq. (2). If the listing price p is equal to or below 0.5, then whether a unit gets sold or not depends on how many buyers and sellers join the platform. Based on the matching mechanism described earlier, a unit will be sold at a price $p \leq 0.5$ if the number of buyers exceed the number of sellers who list the product for p or less, i.e., $d_b \geq d_s \cdot \tilde{F}_s^N(p)$. Because the realizations of D_b and D_s are unknown to the sellers in the no-information model, the selling probability $P^N(p)$ is given by the expected value of the corresponding indication function over D_b and D_s, hence the first line in Eq. (2).

Recall from (1) that any individual seller with reservation price v_s rationally chooses a price p to maximize her expected revenue $p \cdot P^N(p)$ subject to $p \geq v_s$. Thus, the selling probability function $P^N(\cdot)$, given by (2), influences individual sellers' pricing decisions. In the meantime, the pricing decisions of all the sellers, characterized by CDF $\tilde{F}_s^N(\cdot)$ in turn determine the selling probability function $P^N(\cdot)$. This necessitates an equilibrium analysis. *An equilibrium of the sellers' game is reached when no individual seller can improve her expected revenue by unilaterally changing her listing price*, i.e., in equilibrium, Eqs. (1) and (2) must hold simultaneously.

Note that in Model N, the pricing decisions of the sellers whose reservation value $v_s > 0.5$ are irrelevant—because they are not willing to sell the product for a price of 0.5 or less and the buyers are not willing to pay for anything more than 0.5, leading to no successful transactions. Without loss of generality, we assume that these sellers will list and price their products at their reservation values. In the following theorem, we capture the *unique* Pareto-dominant equilibrium for the sellers' game under Model N which maximizes the sum of all sellers' expected revenues among a continuum of equilibria (see the proof of Theorem 1 in the Electronic Companion).

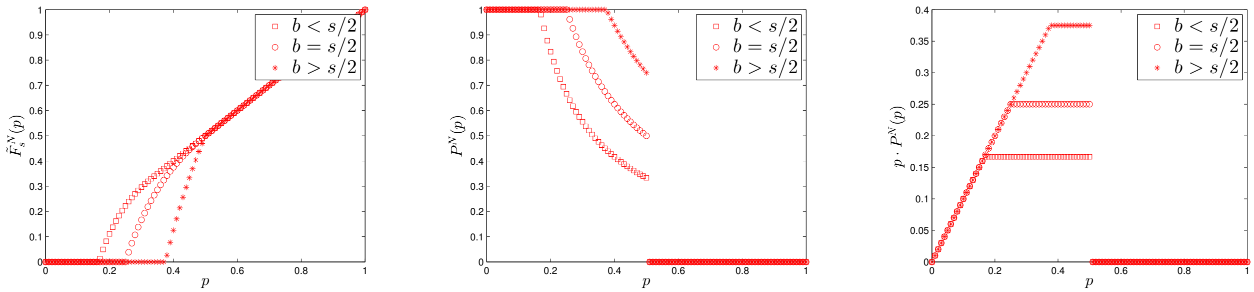
Theorem 1. *When $b \leq s/2$, the sellers' listing price distribution and the selling probability function of the unique Pareto-dominant equilibrium are given as follows:*

$$\tilde{F}_s^N(p) = \begin{cases} 0 & \text{if } 0 \leq p \leq \frac{b}{2s} \\ \frac{2b}{s}\left(1 - \frac{b}{2s \cdot p}\right) & \text{if } \frac{b}{2s} < p \leq \frac{b}{s} \\ p & \text{if } \frac{b}{s} < p \leq 1 \end{cases} \quad \text{and } P^N(p) = \begin{cases} 1 & \text{if } 0 \leq p \leq \frac{b}{2s} \\ \frac{b}{2s \cdot p} & \text{if } \frac{b}{2s} < p \leq 0.5 \\ 0 & \text{if } 0.5 < p < 1 \end{cases}$$

When $b > s/2$, they are given as follows:

$$\tilde{F}_s^N(p) = \begin{cases} 0 & \text{if } 0 \leq p \leq 0.5 - \frac{s}{8b} \\ \frac{2b}{s}\left(1 - \left(0.5 - \frac{s}{8b}\right)\frac{1}{p}\right) & \text{if } 0.5 - \frac{s}{8b} < p \leq 0.5 \\ p & \text{if } 0.5 < p \leq 1 \end{cases} \text{ and } P^N(p) = \begin{cases} 1 & \text{if } 0 \leq p \leq 0.5 - \frac{s}{8b} \\ \left(\frac{1}{2} - \frac{s}{8b}\right)\frac{1}{p} & \text{if } 0.5 - \frac{s}{8b} < p \leq 0.5 \\ 0 & \text{if } 0.5 < p \leq 1 \end{cases}$$

Figure 2 **Illustration of $\tilde{F}_s^N(p)$, $P^N(p)$, and $p \cdot P^N(p)$ as functions of p in panels (a), (b) and (c), respectively.**



(a) (b) (c)

Note. In each panel, we plot three functions based on $b = 1$ and $s \in \{1, 2, 3\}$.

There are a few important observations from the above theorem that we will now discuss in details. An illustration of the results is provided in Figure 2.

First, there exists a threshold $\underline{v}_s (= \frac{b}{2s}$ or $0.5 - \frac{s}{8b})$ in both the $b \le s/2$ case and the $b > s/2$ case such that sellers with reservation values less than $\underline{v}_s$ impose a strictly higher listing price (i.e., no less than $\underline{v}_s$) in equilibrium to achieve revenue maximization. In Figure 2/(a), $\underline{v}_s$ corresponds to the largest p such that $\tilde{F}_s^N(p) = 0$. In particular, $P^N(p) = 1$ for all $p \le \underline{v}_s$ meaning that in equilibrium any product offered at price $\underline{v}_s$ or below will surely get sold to a buyer. However, no seller, even if her reservation value is sufficiently low, has an incentive to lower her price to $\underline{v}_s - \epsilon_1$ where $0 < \epsilon \le \underline{v}_s$, because the expected revenue at $\underline{v}_s - \epsilon$ (i.e., $(\underline{v}_s - \epsilon) \cdot P^N(\underline{v}_s - \epsilon) = \underline{v}_s - \epsilon$), is strictly dominated by the expected revenue at $\underline{v}_s$ (i.e., $\underline{v}_s \cdot P^N(\underline{v}_s) = \underline{v}_s$). In addition, there exists a second threshold $\bar{v}_s (= 0.5$ in Model N) such that all sellers with a reservation value $v_s \le \bar{v}_s$ list the product for a price no higher than $\bar{v}_s$. This implies that any seller with a reservation value $v_s \le \bar{v}_s$ cannot improve her expected revenue by unilaterally choosing a price $\bar{v}_s + \epsilon_2$ for $0 < \epsilon_2 \le 1 - \bar{v}_s$, i.e., $(\bar{v}_s + \epsilon_2) \cdot P^N(\bar{v}_s + \epsilon_2) \le \bar{v}_s \cdot P^N(\bar{v}_s)$. In particular, for Model N, because $P^N(\bar{v}_s + \epsilon_2) = 0$ for $0 < \epsilon_2 \le 1 - \bar{v}_s$, we have $(\bar{v}_s + \epsilon_2) \cdot P^N(\bar{v}_s + \epsilon_2) = 0$, i.e., a seller's expected revenue for a product is zero when it is listed for anything more than 0.5 which is the maximum product price that the buyers are willing to pay for.

Second, we observe that the selling probability function $P^N(p)$, although discontinuous at $p = 0.5$ (because anything more expensive than 0.5 has a zero selling probability), is non-increasing in price; see Figure 2/(b). This is intuitive as a higher listing price for the (same) product corresponds to a lower selling probability. However, the expected revenue, $p \cdot P^N(p)$, as a function of p exhibits the shape of almost a trapezoid (with the right leg truncated), see Figure 2/(c), because p increases while $P^N(p)$ weakly decreases in p. The highest expected revenue occurs at the top base of the trapezoid in Figure 2/(c), which coincides with the interval $[\underline{v}_s, \bar{v}_s]$ of sellers' pricing decisions.

Third, recall that b and s capture the sizes of the buyers' pool and the seller's pool, respectively. As the number of buyers and sellers at the platform are determined by $D_b \sim U[0,b]$ and $D_s \sim U[0,s]$, we can use b and s to mirror the relationship between supply and demand. When $b \le s/2$, because supply far exceeds demand, the sellers have no choice but to list their products at lower prices relative to the opposite case when $b > s/2$ (see Figure 2/(a) where $\tilde{F}_s^N(\cdot)$ for the $b \le s/2$ case is stochastically dominated by that for the $b > s/2$ case). This is due to a fierce competition among the sellers. By contrast, when $b > s/2$, because there is relatively more demand, the sellers are able to list their products for higher prices, and a portion of the sellers even charge a price that is equivalent to the willingness-to-pay of the buyers (i.e., 0.5). These results can further generalize to the following Corollary 1 on the usual stochastic order of $\tilde{F}_s^N(\cdot)$ in b and s. Correspondingly, the selling probability function $P^N(\cdot)$ increases from the case of $b \le s/2$ to the case of $b > s/2$, see Figure 2/(b), and so does the expected revenue for the sellers (whose $v_s \le \bar{v}_s$), see Figure 2/(c). All of these are driven by the market force governed by b and s.

Corollary 1. *Equilibrium listing prices (stochastically) increase in b and decrease in s, i.e., $Pr(p > x) = 1 - \tilde{F}_s^N(x)$ weakly increases in b and decreases in s for all $x \in [0,1]$.*

Finally, we point out that the proof of Theorem 1 is by no means straightforward. We provide the complete proof of Theorem 1 together with proofs for all other results in the Electronic Companion to the paper.

4.1. Performance Metrics

In this subsection, we study the profitability of the platform in our model by considering two performance metrics, namely sales (i.e., total dollar amount of all transactions) and sales volume (i.e., total number of transactions). More details on why they are of interest to us have been given in §1 and will also be highlighted in §6.

In equilibrium, the sellers list their products according to prices determined by $\tilde{F}_s^N(\cdot)$, and the transactions between the sellers and the buyers occur in the increasing order of the listing prices (from the lowest listing price to the highest). Let us denote $p_{max}^N(d_b, d_s)$ as the maximum transaction price given d_b and d_s, i.e., the highest listing price at which a successful transaction occurs. For notational convenience, we write $p_{max}^{(\cdot)}(d_b, d_s)$ simply as $p_{max}^{(\cdot)}$ in the remaining paper whenever doing so is not causing confusion. Two scenarios need to be discussed: (i) If buyers are outnumbered by sellers whose listing price is no higher than the buyers' product valuation 0.5 (i.e., if $d_b \le d_s \cdot \tilde{F}_s^N(0.5)$), $p_{max}^N(\le 0.5)$ can be determined uniquely by solving $d_b = d_s \cdot \tilde{F}_s^N(p_{max}^N)$; (ii) otherwise, (i.e., if $d_b > d_s \cdot \tilde{F}_s^N(0.5)$), there are sufficient buyers to acquire all products that are listed for 0.5 or below, and hence $p_{max}^N = 0.5$.

From Theorem 1, it can be verified that $\tilde{F}_s^N(\cdot)$ is differentiable everywhere except at $\underline{v}_s$ and $\bar{v}_s$. Let $\tilde{f}_s^N(\cdot)$ denote the corresponding probability density function (PDF). Then, the *expected sales volume* in equilibrium, denoted by Q, is given by

$$Q^N \doteq \iint\limits_{D_b, D_s} q^N(d_b, d_s) dD_b dD_s \text{ where } q^N(d_b, d_s) \doteq \begin{cases} d_b = d_s \cdot \tilde{F}_s^N(p_{max}^N) & \text{if } d_b \leq d_s \cdot \tilde{F}_s^N(0.5); \\ d_s \cdot \tilde{F}_s^N(0.5) & \text{if } d_b > d_s \cdot \tilde{F}_s^N(0.5). \end{cases}$$

And the *expected sales* in equilibrium, denoted by T, can be characterized by

$$T^N \doteq \iint\limits_{D_b, D_s} t^N(d_b, d_s) dD_b dD_s \text{ where } t^N(d_b, d_s) \doteq \begin{cases} d_s \cdot \int_0^{p_{max}^N} x \cdot \tilde{f}_s^N(x)\, dx & \text{if } d_b \leq d_s \cdot \tilde{F}_s^N(0.5); \\ d_s \cdot \int_0^{0.5} x \cdot \tilde{f}_s^N(x)\, dx & \text{if } d_b > d_s \cdot \tilde{F}_s^N(0.5). \end{cases}$$

In the next result, we establish the impact of the sizes of the buyers' pool (b) and the sellers' pool (s) on Q^N and T^N.

PROPOSITION 1. *Under the no-information model,*

(i). The expected sales volume Q^N increases in b and s;

(ii). The expected sales T^N increase in b and are unimodal in s.

Proposition 1/(i) is in line with expectations, given that when there is either more demand or more supply to the platform, which is essentially a two-sided market, more transactions will occur leading to a higher sales volume.

The more interesting result is Proposition 1/(ii) which states that sales are monotonic in the size of the buyers' pool but not in the size of the sellers' pool. Intuitively, when there are more buyers in the market (i.e., an increased b), equilibrium prices (Corollary 1) as well as the sales volume goes up (Proposition 1/(i)), thus higher sales is expected as sales can be viewed roughly as the multiplication of price and sales volume. Yet, when there are more sellers in the market (i.e., an increased s), there are two competing forces which effect the sales. On one hand, equilibrium prices go down (Corollary 1) due to increased competition among the sellers, and on the other hand, sales volume picks up (Proposition 1/(i)). These two opposing effects lead to the non-monotonic behavior of the sales with respect to s which first increases then decreases. It may also be helpful to think of the limiting cases: (i) As $s \to 0$, there are no sellers in the platform so no sales occur; (2) as $s \to \infty$, excessive sellers drive the listing/sales prices down towards zero, again yielding near zero dollar sales for the platform despite a significant sales volume. The important managerial implication here is that although increases in the size of the buyers' pool should always be welcomed by the platform, increases in the size of the sellers' pool should be treated cautiously to avoid unintended consequences on the sales.

5. Informational Models

In this section, we will investigate how product information and/or market information influence the sellers' pricing decisions in equilibrium. Note that the impact of different information structures on the sales and sale volume of the platform will be examined separately in §6. To this end, we will propose and analyze three informational models (when $I_b = 1$ or/and $I_s = 1$): the production-information model in §5.1, the market-information model in §5.2, and the product-and-market-information model in §5.3.

5.1. Product-Information Model

We start with the product-information(-only) model, or Model P, where the platform provides the product information but not the market information (i.e., $I_b = 1$ and $I_s = 0$). Specifically, with the product information being provided, the buyers are able to accurately assess the product and formulate an individual-specific product valuation. Therefore, different from Model N in §4 where all buyers evaluated the product with the expected value 0.5, the buyers under Model P exhibit heterogeneous product valuations, ranging uniformly from 0 to 1.

Let $\tilde{F}_s^P(\cdot)$ denote the CDF of the equilibrium listing price distribution of the sellers in Model P (superscript P represents the product-information setting). The selling probability function, given earlier in (2) for Model N, needs to be updated to a general expression for Model P:

$$P^P(p) \doteq \mathop{\mathbb{E}}_{D_s, D_b}\left[\mathbb{1}\left\{D_b(1 - F_b(p)) \geq D_s \cdot \tilde{F}_s^P(p)\right\}\right]. \tag{3}$$

Eq. (3) reflects the notion that a product at price p gets sold, upon the realization of $D_b = d_b$ and $D_s = d_s$, if and only if the number of buyers whose valuation for the product exceeds p is greater than or equal to the number of sellers who list the product for p or less, i.e., $d_b(1 - F_b(p)) \geq d_s \cdot \tilde{F}_s^P(p)$.

Following similar techniques used in the proof for Theorem 1, we can obtain the equilibrium listing price distribution and the corresponding selling probability function for Model P.

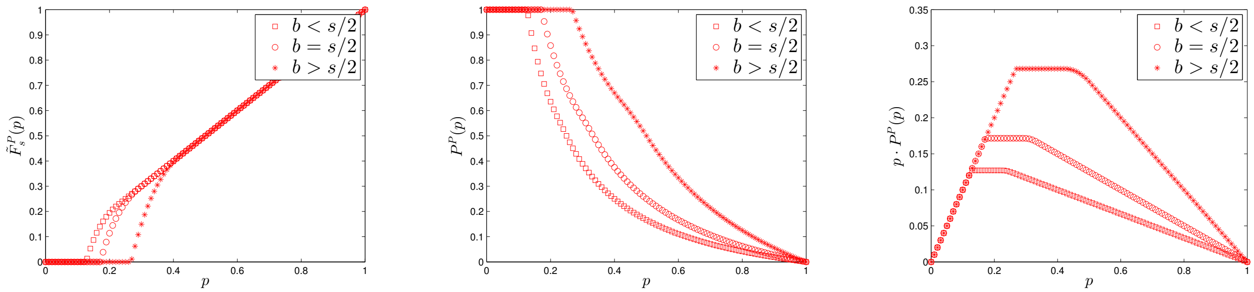
THEOREM 2. *Under Model P, the sellers' listing price distribution and the selling probability function of the unique Pareto-dominant equilibrium are given as follows:*

$$\tilde{F}_s^P(p) = \begin{cases} 0 & \text{if } 0 \leq p \leq \frac{b+s-\sqrt{(2b+s)s}}{b} \\ \frac{2b}{s}(1-p)\left(1 - \frac{b+s-\sqrt{(2b+s)s}}{b \cdot p}\right) & \text{if } \frac{b+s-\sqrt{(2b+s)s}}{b} < p \leq 1 - \sqrt{\frac{s}{2b+s}} \\ p & \text{if } 1 - \sqrt{\frac{s}{2b+s}} < p \leq 1 \end{cases}$$

and

$$P^P(p) = \begin{cases} 1 & \text{if } 0 \leq p \leq \frac{b+s-\sqrt{(2b+s)s}}{b} \\ \frac{b+s-\sqrt{(2b+s)s}}{b}\frac{1}{p} & \text{if } \frac{b+s-\sqrt{(2b+s)s}}{b} < p \leq 1 - \sqrt{\frac{s}{2b+s}} \\ 1 - \frac{s}{2b}\frac{p}{1-p} & \text{if } 1 - \sqrt{\frac{s}{2b+s}} < p \leq \frac{b}{b+s} \\ \frac{b}{2s}\frac{1-p}{p} & \text{if } \frac{b}{b+s} < p \leq 1 \end{cases}$$

Figure 3 **Illustration of $\tilde{F}_s^P(p)$, $P^P(p)$, and $p \cdot P^P(p)$ as functions of p in panels (a), (b) and (c), respectively.**



(a) (b) (c)

Note. In each panel, we plot three functions based on $b = 1$ and $s \in \{1, 2, 3\}$.

The above result is illustrated in Figure 3. Similar to Model N, we observe that the sellers charge higher prices in equilibrium when the size of the buyers' pool (b) increases or the size of the sellers' pool (s) decreases; see Figure 3/(a). This result is formalized as follows.

Corollary 1 is preserved under Model P, i.e., equilibrium listing prices under Model P (stochastically) increase in b and decrease in s.

On the other hand, the selling probability function $P^P(p)$ is now positive for all $p \in [0, 1)$ under Model P, compared to Model N where the selling probability is zero for $p > 0.5$; see Figure 3/(b) cf. Figure 2/(b). Correspondingly, the expected revenue, $p \cdot P^N(p)$, as a function of p exhibits the shape of a full trapezoid, see Figure 3/(c). This implies that sellers with reservation values $v_s > 0.5$ receive a higher expected revenue under Model P (positive amount) compared to under Model N (zero). Therefore, provision of the product information benefits the sellers with high reservation values because now there are high-valuation buyers who can afford the listing prices of these sellers.

Finally, given the sellers' pricing decisions characterized by CDF $\tilde{F}_s^P(\cdot)$ and the buyers' product valuation $v_b \sim U[0, 1]$, let $p_{max}^P(d_b, d_s)$ denote the maximum transaction price between the two sides when $D_b = d_b$ and $D_s = d_s$. Then, $p_{max}^P(d_b, d_s)$ can be uniquely determined by solving $d_b[1 - F_b(p_{max}^N)] = d_s \cdot \tilde{F}_s^N(p_{max}^N)$. The total expected sales volume and sales for the platform under Model P can be derived as follows:

$$Q^P \doteq \iint_{D_b, D_s} q^P(d_b, d_s) dD_b dD_s \text{ where } q^P(d_b, d_s) \doteq d_s \cdot \tilde{F}_s^P(p_{max}^P);$$

$$T^P \doteq \iint_{D_b, D_s} t^P(d_b, d_s) dD_b dD_s \text{ where } t^P(d_b, d_s) \doteq d_s \cdot \int_0^{p_{max}^P} x \cdot \tilde{f}_s^P(x)\, dx.$$

Furthermore, we find that the impact of b and s on the platform's expected sales and sales volume under Model P remains qualitatively the same as in Model N, i.e., *Proposition* 1 *is preserved under Model* P.

As the primary purpose for this section is to establish the sellers' pricing decisions in equilibrium, we will postpone the comparison of the expected sales and sales volume between Model P and Model N (or between any two models) to §6 where we discuss the impact of different information structures on the profitability of the platform.

5.2. Market-Information Model

Now, we will introduce the third model, the market-information(-only) model, or Model M, in which the platform provides the market information but not the product information (i.e., $I_b = 0$ and $I_s = 1$). It means that the sellers know the realized market size $D_b = d_b$ and $D_s = d_s$ when making pricing decisions. Yet, without the product information, the sellers also understand that all of the buyers will valuate the product at its mean value, i.e., $v_b = \mathbb{E}(U[0,1]) = 0.5$.

Such an information structure substantially simplifies the seller's game. In particular, denote $p^M_{max}(d_b, d_s)$ as the maximum transaction price between the sellers and buyers for given d_b and d_s. Then, based on the matching mechanism described in §3, in equilibrium, all the sellers whose reservation value $v_s \le p^M_{max}$ should list their products at the price p^M_{max} in order to maximize their expected revenue, and the corresponding products will be sold to the buyers with valuation $v_b \ge p^M_{max}$. Moreover, the number of sellers who successfully sell a product must be equal to the number of buyers who successfully buy a product, i.e.,

$$d_b \cdot [1 - F_b(p^M_{max})] = d_s \cdot F_s(p^M_{max}). \tag{4}$$

We characterize the sellers' listing price distribution and selling probability function of the unique (Pareto-dominant) equilibrium in the following theorem. Note that the pricing decisions of the sellers with a reservation value $v_s > p^M_{max}$ are irrelevant so without loss of generality we assume that their listing prices are the same as their reservation values.

THEOREM 3. *Under Model M, given the market information (d_b, d_s), the sellers' listing price distribution and the selling probability function in equilibrium are as follows:*

$$\tilde{F}^M_s(p|d_b, d_s) = \begin{cases} 0 \ \textit{if}\ 0 \le p < p^M_{max}(d_b, d_s) \\ p \ \textit{if}\ p^M_{max}(d_b, d_s) \le p \le 1 \end{cases} \textit{and } P^M(p|d_b, d_s) = \begin{cases} 1 \ \textit{if}\ 0 \le p \le p^M_{max}(d_b, d_s) \\ 0 \ \textit{if}\ p^M_{max}(d_b, d_s) < p \le 1 \end{cases}$$

where by (4),

$$p^M_{max}(d_b, d_s) \doteq \begin{cases} d_b/d_s & \textit{if}\ d_b < d_s/2; \\ 0.5 & \textit{if}\ d_b \ge d_s/2. \end{cases} \tag{5}$$

It is clear from (5) that $p^M_{max}(d_b, d_s)$ weakly increases in d_b and decreases in d_s. Therefore, we can immediately deduce the following result:

Corollary 2. *Equilibrium listing prices (stochastically) increase in d_b and decrease in d_s, i.e., $Pr(p > x|d_b, d_s) = 1 - \tilde{F}_s^M(x|d_b, d_s)$ weakly increases in d_b and decreases in d_s for all $x \in [0, 1]$.*

Essentially, Corollary 2 for Model M is analogous to Corollary 1 for Model N, both of which state the notion that the product prices are higher in equilibrium when there is more demand but lower when there is more supply, due to the market force. However, in Model M, because the market information is provided to the buyers and sellers, the monotonic result is based on realized market size d_b and d_s, instead of market parameters b and s as in Model N when no market information is provided by the platform.

From Theorem 3, we can further derive the sales volume and the sales for the platform for any given market size d_b and d_s, denoted by $q^M(d_b, d_s)$ and $t^M(d_b, d_s)$, respectively. Hence, we can characterize the expected sales volume Q^M and expected sales T^M as follows:

$$Q^M \doteq \iint\limits_{D_b, D_s} q^M(d_b, d_s) dD_b dD_s \text{ where } q^M(d_b, d_s) \doteq \begin{cases} d_b & \text{if } d_b < d_s/2; \\ d_s/2 & \text{if } d_b \geq d_s/2; \end{cases}$$

$$T^M \doteq \iint\limits_{D_b, D_s} t^M(d_b, d_s) dD_b dD_s \text{ where } t^M(d_b, d_s) \doteq p_{max}^M(d_b, d_s) \cdot q^M(d_b, d_s) = \begin{cases} d_b^2/d_s & \text{if } d_b < d_s/2; \\ d_s/4 & \text{if } d_b \geq d_s/2. \end{cases}$$

It can be verified that the impact of the sizes of the buyers' pool (b) and the sellers' pool (s) on the platform's expected sales and sales volume remains unchanged from previous models, i.e.,

Proposition 1 *is preserved under Model M.*

5.3. Product-and-Market-Information Model

In the last model, we will analyze the scenario where the online marketplace provides both the product and the market information on its platform (i.e., $I_b = 1, I_s = 1$), which we will refer to as Model B. The thought process of the equilibrium analysis is similar to that of Model M. In particular, if we use $p_{max}^B(d_b, d_s)$ to denote the maximum transaction price between the sellers and buyers given d_b and d_s, then in equilibrium, all sellers with a reservation value $v_s \leq p_{max}^B$ will list their products at the price p_{max}^B, and the corresponding products will be sold to buyers with valuation $v_b \geq p_{max}^B$. Similar to Eq. (4), we have

$$d_b \cdot [1 - F_b(p_{max}^B)] = d_s \cdot F_s(p_{max}^B). \tag{6}$$

We characterize the sellers' listing price distribution and selling probability function of the (Pareto-dominant) equilibrium in the following theorem. Note that the pricing decisions of the sellers with a reservation value $v_s > p_{max}^B$ are again irrelevant so without loss of generality we can assume that their listing price is the same as reservation value.

THEOREM 4. *Under Model B, given the market information (d_b, d_s), the sellers' listing price distribution and the selling probability function in equilibrium are as follows:*

$$\tilde{F}_s^B(p|d_b,d_s) = \begin{cases} 0 \ \text{if} \ 0 \le p < p_{max}^B(d_b,d_s) \\ p \ \text{if} \ p_{max}^B(d_b,d_s) \le p \le 1 \end{cases} \text{ and } P^B(p|d_b,d_s) = \begin{cases} 1 \ \text{if} \ 0 \le p \le p_{max}^B(d_b,d_s) \\ 0 \ \text{if} \ p_{max}^B(d_b,d_s) < p \le 1 \end{cases}$$

where by (6), $p_{max}^B(d_b,d_s) \doteq d_b/(d_b+d_s)$.

As $p_{max}^B(d_b,d_s)$ increases in d_b and decreases in d_s, we can easily verify that the sellers' equilibrium listing prices given in Theorem 4 (stochastically) increase in d_b and decrease in d_s, i.e.,

Corollary 2 is preserved under Model B.

From Theorem 4, we can also obtain expressions for the platform's expected sales volume and expected sales under Model B as follows:

$$Q^B \doteq \iint_{D_b,D_s} q^B(d_b,d_s)dD_b dD_s \text{ where } q^B(d_b,d_s) \doteq d_s \cdot F(p_{max}^B) = \frac{d_b \cdot d_s}{d_b+d_s};$$

$$T^B \doteq \iint_{D_b,D_s} t^B(d_b,d_s)dD_b dD_s \text{ where } t^B(d_b,d_s) \doteq p_{max}^B(d_b,d_s) \cdot q^B(d_b,d_s) = \frac{d_b^2 \cdot d_s}{(d_b+d_s)^2}.$$

It can be further verified that the impact of b and s on the platform's expected sales and sales volume remains unchanged from before, i.e.,

Proposition 1 *is preserved under Model B.*

6. Optimal Information Structure for the Platform

Having characterized the Pareto-dominant equilibrium for each of the four information structures in §4 and §5 (i.e., Model N, Model P, Model M and Model B), we will now seek an answer in this section to the fundamental question of the paper: Should the platform provide the buyers and sellers with the product and/or market information?

To this end, recall that in §3 we have distinguished two types of objective functions for the platform. This is because, in practice, the platform earns profit by either charging a fixed commission per transaction or a proportional commission as a percentage of the sales. When the platform practices the fixed-commission scheme (e.g., Swappa), its revenue increases proportionally to the sales volume (at the rate of the fixed fee). In comparison, when the platform practices the proportional-commission scheme (e.g., eBay), its revenue increases proportionally to the total sales (at the rate of the commission rate). We consider the profitability of the platform by focusing on its sales and sales volume as the two metrics. In particular, we investigate the optimal information structure for the fixed-commission scheme (i.e., the sales volume maximization) in §6.1, and the proportional-commission scheme (i.e., the sales maximization) in §6.2.

6.1. Fixed-Commission Scheme

Under the fixed-commission scheme, the platform seeks to maximize the sales volume in the marketplace. By comparing the expected sales volume, Q, under all four information structures, we derive the following result:

THEOREM 5. *The expected sales volumes satisfy the following relations: (i). $Q^N = Q^M$; (ii). $Q^B \geq Q^P$; (iii). $Q^B \geq Q^M$ if and only if $b/s > \theta_1 \doteq 1.0624$.*

Theorem 5/(i) reveals that in the absence of the product information (i.e., $I_b = 0$), provision of the market information has no impact on the sales volume of the platform (i.e., $Q^N = Q^M$). Recall that without the product information, the buyers cannot accurately assess whether the product will meet their individual needs and, therefore, evaluate the product with an expected value. As a result, based on the realized market size of the buyers and sellers (d_b and d_s), either all of the d_b buyers are able to acquire the product or all of the $d_s \cdot F_s(0.5)$ sellers whose reservation value is no higher than the buyers' mean valuation 0.5 are able to sell their products, i.e., $q^N = q^M = \min\{d_b, d_s \cdot F_s(0.5)\}$. In other words, the lesser of the two quantities, which are both independent of the market information revelation, determines the sales volume of the platform. Hence, providing the market information has no impact on the platform's expected sales volume in the absence of the product information. Yet, it is worth mentioning here that provision of the market information by the platform does affect the sellers' optimal pricing decisions and thus the expected sales of the platform, as we should later see in §6.2.

By contrast, if the platform is already providing the product information to the buyers and sellers (i.e., $I_b = 1$), then Theorem 5/(ii) states that it is beneficial for the platform to also provide the market information (i.e., $Q^B \geq Q^P$). Note that in the presence of the product information, the buyers are able to infer their exact valuations of the product, which creates heterogeneous product valuations among them and incentivizes the sellers to enlarge their price differentiation. However, the increased spread of the listing price distribution could create mismatch between the supply and the demand, and this is where the market information can add its value. When given the market information, the sellers could adaptively adjust their prices in response to the realized market size, and demand becomes better matched with supply. Therefore, in this case, provision of the market information leads to a higher sales volume for the platform, as it facilitates more transactions to occur given that the sellers are able to make target pricing decisions in response to the realized market size.

Yet, different from the market information decision, whether the platform should provide the product information is a less straightforward decision. In particular, we can see from Theorem 5/(iii) that the product information decision is contingent on the ratio of b/s. As b and s correspond to the sizes of the buyers' pool and the sellers' pool, respectively, and the realized market size of the buyers and the sellers are uniformly distributed, we will address the ratio of b/s simply as the *demand-to-supply ratio*. Borrowing a term from the real estate industry, we will categorize a market with the demand-to-supply ratio greater than one (resp. less than one) as a seller's market (resp. a buyer's market) in the remaining paper. Theorem 5/(iii) reveals that when the demand-to-supply ratio exceeds $\theta_1 = 1.0624$ (i.e., approximately in a seller's market), it is optimal for the platform to supply the product information; otherwise, it should not supply the product information.

To intuit the result of Theorem 5/(iii), let us first consider a seller's market (i.e., the market is under a high demand-to-supply ratio). In a seller's market, the buyers outnumber the sellers and if there were no product information, all the buyers can only afford low or median product price listings (i.e., the buyers are willing to pay no more than the mean valuation 0.5). Hence, the sales volume is constrained by the sellers who have high reservation values (and their listing prices are even higher). Under this scenario, it is optimal for the platform to reveal the product information, as doing so increases the spread of product valuations for buyers, with those buyers that have high product valuations willing to pay high prices, leading to more sales. On the other hand, in a buyer's market (i.e., the market is under a low demand-to-supply ratio), there are more sellers than buyers. Buyers with median or high product valuations can always acquire the product, but buyers with low product valuations may not be able to because they are the last ones to be matched with the sellers in the marketplace and these remaining sellers may not have low listing prices for the product. In this case, the sales volume of the platform is restricted by the number of buyers who have low product valuations. Therefore, by not sharing the product information, the platform can reduce the number of buyers with low product valuations and improve overall sales volume.

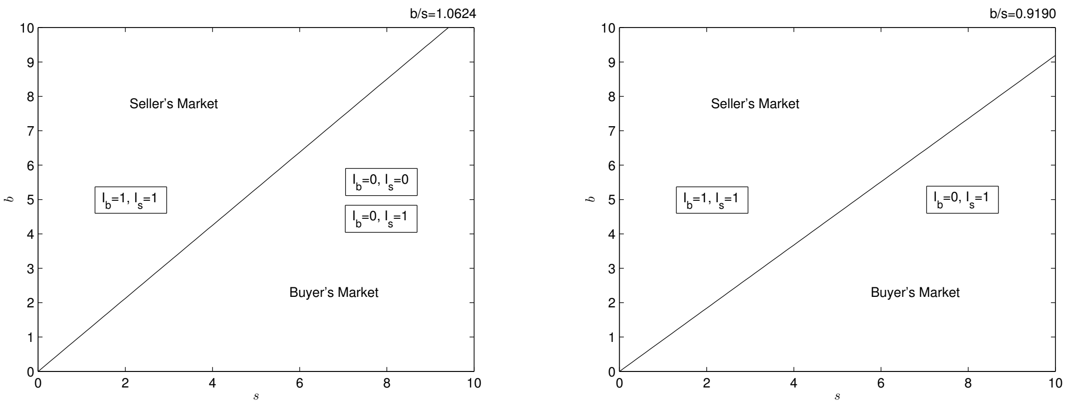
Based on Theorem 5, we can summarize the platform's optimal information strategy under the fixed-commission scheme (i.e., maximization of expected sales volume) in the following proposition, which is also illustrated by Figure 4/(a).

PROPOSITION 2. *Under the fixed-commission scheme, it is optimal for the platform to provide both the product and market information when $b/s > \theta_1$. Otherwise, the platform is in its best interest not to provide the product information. Without the provision of product information, market information does not add value to the profitability of the platform.*

6.2. Proportional-Commission Scheme

Under the proportional-commission scheme, the platform seeks to maximize its expected total sales T instead. By comparing the expected total sales under different information structures, we obtain the following result:

Figure 4 **Optimal information structure for the platform under different commission schemes.**



(a) Fixed-Commission Scheme

(b) Proportional-Commission Scheme

Theorem 6. *The expected sales satisfy the following relations: (i). $T^M \geq T^N$; (ii). $T^B \geq T^M$ if and only if $b/s > \theta_2 \doteq 0.9190$; (iii). $T^B \geq T^N$ if and only if $b/s < \theta_3 \doteq 0.3230$ or $b/s > \theta_4 \doteq 0.7304$.*

Akin to Theorem 5 for sales volume maximization, Theorem 6 suggests that the optimal information structure for the platform, when the goal is to maximize the sales, is contingent exclusively on the demand-to-supply ratio b/s. In what follows, we will elaborate on each part of Theorem 6.

First, Theorem 6/(i) reveals that in the absence of the product information (i.e., $I_b = 0$), provision of the market information always improves the total sales. This draws a direct contrast to its counterpart, Theorem 5/(i), which indicates that provision of the market information has no impact on the sales volume when no product information is provided by the platform. The intuition for Theorem 6/(i) is that even if the same number of transactions occur between the sellers and buyers with or without market information, when there is market information, the sellers can make target pricing decisions in response to the realized market size, boosting the transaction prices.

Second, if the platform releases the market information (i.e., I_s=1), Theorem 6/(ii) suggests that the platform can improve its expected total sales by releasing the product information if and only if $b/s > \theta_3 = 0.9190$, i.e., approximately in a seller's market. The intuition here is similar to that of Theorem 5/(iii). Specifically, in a seller's market (i.e., the market is under high demand-to-supply ratio) without the product information, the buyers will only pay low or median prices for the product, and the total sales volume will mainly be driven by how many products from the high-reservation-value/high-price sellers can be acquired. By providing the product information, the platform facilitates the buyers to develop heterogeneous product valuations, and the high-reservation-value sellers, who could not be matched with buyers before, can now find buyers, boosting the total sales for the platform. On the other hand, in a buyer's market (i.e., the market is under a low demand-to-supply ratio) without the product information, the buyers can always successfully acquire the product without paying a high price. However, if the platform provides the product information, buyers with low product valuations may return empty-handed because they will be the last ones to acquire the product according to the matching mechanism. In this case, the platform can benefit from not providing the product information to avoid such lost sales.

Finally, Theorem 6/(iii) states that providing both the product and the market information leads to higher total sales for the platform compared to the strategy of withholding both types of information, if and only if the demand-to-supply ratio is either sufficiently small or sufficiently large but not when it is at an intermediate level (i.e., if and only if $b/s < \theta_3$ or $b/s > \theta_4$). To intuit the result, let us first consider the effect of product information and the effect of market information on sales in isolation. On one hand, the provision of product information reduces (resp. increases) sales when the demand-to-supply ratio is under (resp. above) some threshold, see the explanation for Theorem 6/(ii). On the other hand, the provision of market information always increases sales, see the explanation for Theorem 6/(i). When the demand-to-supply ratio is sufficiently small (i.e., $b/s < \theta_3$), the negative effect on sales by providing product information is outweighed by the positive effect on sales by providing market information, making announcing both types of information a better strategy than withholding them. However, when the demand-to-supply ratio is at an intermediate level (i.e., $b/s \in [\theta_3, \theta_4]$), the negative effect on sales by providing product information can no longer be offset by the positive effect on sales by providing market information, making announcing both types of information a worse strategy than announcing them. Last, when the demand-to-supply ratio becomes sufficiently high (i.e., $b/s > \theta_4$), the provision of either product information or market information has a positive effect on sales. Combining both positive effects, the platform is certainly better off with providing both types of information compared to not providing any information.

It is necessary to note now that Theorem 6 does not present any comparison results on the expected total sales under Model P, i.e., T^P. This is because T^P cannot be explicitly characterized and analytically compared to the other models. Nevertheless, through extensive numerical experiments that we have conducted, we find that Model B continues to dominate Model P for the expected total sales, i.e., $T^B \geq T^P$ (we showed $Q^B \geq Q^P$ analytically in Theorem 5/(ii)). Accordingly, we can infer the platform's optimal information strategy under the proportional-commission scheme, when its goal is to maximize expected total sales. The result is presented in the following proposition and illustrated in Figure 4/(b).

Proposition 3. *Under the proportional-commission scheme, it is optimal for the platform to provide both the product and market information when $b/s > \theta_2$. Otherwise, it is optimal for the platform to provide only the market information.*

By comparing Proposition 2 and Proposition 3 (Figure 4/(a) and (b) accordingly), we can discern the similarities and the differences concerning the optimal information structure/strategy for the platform when the platform practices a fixed or a proportional-commission scheme. Under either scheme, it is only optimal for the platform to broadcast both the product and market information in a seller's market which facilitates a better matching between the sellers' reservation values and the buyers' actual valuations for the product. By contrast, when facing a buyer's market, the platform should always conceal the product information so that the buyers' valuations for the product are less dispersed. In addition, in a buyer's market, even though providing only the market information is an optimal strategy for the platform regardless of the underlying commission scheme, the market information will add actual value only for the fixed-commission scheme but not the proportional-commission scheme.

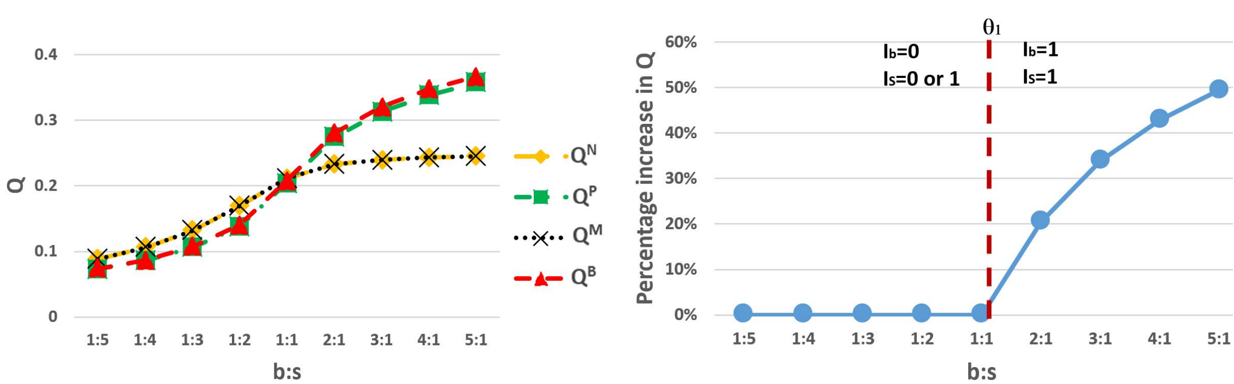
7. Value of Information

In this section, we will quantitatively assess the benefits of the provision of information for the platform based on the optimal information structure identified in §6. We will start with the scenario where the platform practices the fixed-commission scheme, i.e., the objective of the platform is to maximize the expected sales volume.

In Figure 5/(a), we plot the expected sales volume under the four different information structures (i.e., Model N, Model P, Model M, and Model B) for various demand-to-supply ratios (i.e., b/s). As the optimal information structure depends exclusively on the ratio of b/s, without loss of generality, we set $s=1$ and vary the value of b to make the plot. We observe that in all four models the sales volume increases in b, which is consistent with the theoretical prediction of Proposition 1/(i).

To gauge the potential benefits of the provision of information, we benchmark the sales volume under the optimal information structure to that of the no-information structure. We define the *value*

Figure 5 Sales volume and the corresponding Value of Information under the fixed-commission scheme



(a) Sales Volume

(b) Value of Information (for Sales Volume) *of information* for the sales volume, denoted by VOI^Q, as the percentage increase in the expected sales volume between the two, i.e., $VOI^Q := \max\{Q^N, Q^P, Q^M, Q^B\}/Q^N - 1$. For example, when the optimal information structure is the no-information case itself, we have $\max\{Q^N, Q^P, Q^M, Q^B\} = Q^N$ and $VOI^Q = 0\%$, i.e., there is no value to the platform in obtaining and providing the product or the market information.

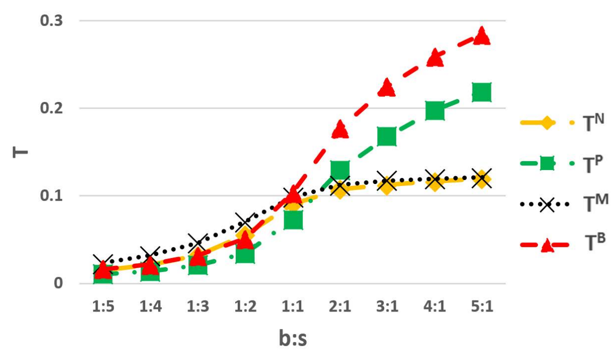
We plot VOI^Q for various demand-to-supply ratios b/s in Figure 5/(b). Immediately, we observe that more information does not always benefit the platform. In particular, consistent with Proposition 2, we see that when the demand-to-supply ratio is less than the threshold $\theta_1 = 1.0624$, the platform receives no value (or even negative value) in providing information. Yet, as the demand-to-supply ratio increases with more buyers, the market moves into a seller's market, in which case the platform becomes incentivized to provide both the product and market information in order to maximize the sales volume (i.e., in this case Model B is the optimal information structure for the platform).

We further observe that the value of information increases in the demand-to-supply ratio from 0% when $b/s = 1:1$ to approximately 50% when $b/s = 5:1$, see Figure 5/(b). This observation suggests that when the platform grows mature and starts to attract more and more buyers, investing in a review system (to provide detailed product information to buyers) as well as website visitor tracking technology (to provide real-time information on market size to sellers) can be a wise strategy for the platform which will generate increasing returns as the value of information grows.

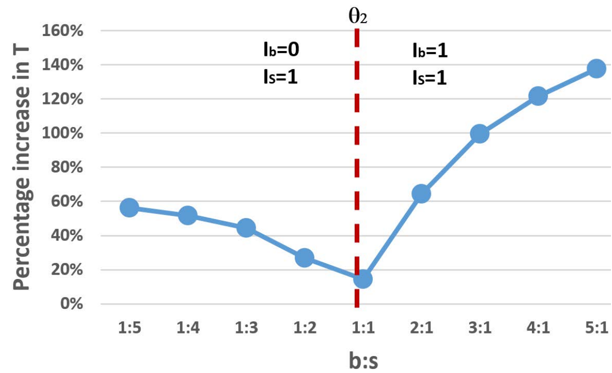
Next, we turn to a platform that practices the proportional-commission scheme. Because the goal of the platform becomes maximizing the expected total sales, we define the *value of information* for the sales, denoted by VOI^T, as the percentage increase in sales under the optimal information structure over the no-information structure, i.e., $VOI^T := \max\{T^N, T^P, T^M, T^B\}/T^N - 1$. In Figure 6/(a) and (b), we plot the expected total sales under the four information structures studied in the paper and the value of information for sales, respectively, as a function of the demand-to-supply ratio b/s. We continue to set $s = 1$ and vary the value of b for these plots.

There are some interesting observations that arise from Figure 6. First, the expected sales in all models increase in b, which is consistent with the theoretical prediction of Proposition 1/(ii). It is worth noting here that if we fixed b and let s vary, then the sales will be unimodal in s. Yet, the value of information will remain unchanged as long as the value of b/s stays the same. Second, the value of information for the sales, intriguingly, exhibits a non-monotonic pattern with respect to the demand-to-supply ratio b/s. Specifically, additional information can help improve sales when the platform is faced with either a seller's market (i.e., $b/s \leq 0.9705$) or a buyer's market (i.e., $b/s > 0.9705$), which is consistent with Proposition 3. Furthermore, we observe that the value of information increases in the intensity of either market. Hence, we can conclude that

Figure 6 **Total sales and the corresponding Value of Information under the proportional-commission scheme**



(a) Sales



(b) Value of Information (for Sales)

a revenue-maximizing platform can substantially benefit from having and releasing information when the market is unbalanced, meaning that either sellers greatly outnumber buyers or vice versa. In particular, as we can see in Figure 6/(b), as the ratio of b/s approaches $5:1$, the value of information reaches approximately 140%. The takeaway here is that for a buyer-popular revenue-maximizing online marketplace, it is particularly advantageous if the platform can obtain and use both product and market information correctly.

Finally, by comparing Figure 6/(b) to Figure 5/(b), we observe that information is more valuable to the platform when the goal of the platform is to maximize sales rather than sales volume, and when it faces a seller's market rather than a buyer's market.

8. Concluding Remarks

We have studied the role and the value of information in an online marketplace in this paper. We find that information could be a double-edged sword for the platform. On one hand, the platform benefits from the revelation of all (product and market) information in a seller's market. On the other hand, it is less helpful or even harmful for the platform to have all information revealed in a buyer's market.

In particular, product information should be only provided in a seller's market and not in a buyer's market, regardless of whether the goal of the platform is to maximize sales or sales volume. This is because in a seller's market, there are more buyers than sellers. If all sellers can sell their products, the platform can maximize sales or sales volume, especially when those sellers with high reservation value can sell their products. Provision of product information is a great strategy in this case as it allows the buyers to develop heterogeneous product valuations with those buyers that have higher product valuations willing to pay for higher prices. In contrast, in a buyer's market, sellers outnumber buyers, and total sales or sales volume of the platform can be maximized if all buyers can successfully purchase the product. By not providing the product information, the platform benefits from customers' product valuations being less dispersed, meaning that there are fewer customers with low willingness-to-pay for the product.

We also find that while market information typically helps the platform improve sales, if the goal is to increase sales volume, it may not add real value to the platform. This can happen, for example, when product information is not provided to buyers and sellers. In this case, the total sales volume depends exclusively on the number of buyers and sellers who join the platform but does not depend on the revelation of the market information.

Our other contribution in this research is the underlying technical analysis which identifies the (Pareto-dominant) equilibrium of the sellers' pricing decisions under various informational structures. As noted by Chen et al. (2016) on the seller competition for an online marketplace (see p. 591 of the paper), "it is difficult to make a comprehensive and rigorous analysis for the general seller competition case," and we have sought to carry out the analysis under some reasonable assumptions. We find that after taking into account sellers' strategic pricing decisions, the sales of the online marketplace increase in the size of the buyers' pool but are actually unimodal in the size of the sellers' pool. This is because the online marketplace is essentially a two-sided market where the cross-side network effect is positive (buyers prefer more sellers and vice versa) but same-side network effect is negative (each buyer prefers less other buyers to drive down equilibrium prices and each seller prefers less other sellers to drive up equilibrium prices). Hence, while a larger pool of buyers always results in more sales for the platform (more transactions plus higher transaction prices), a larger pool of sellers does not (more transactions yet lower prices).

As the objective of our model framework is to capture the essential elements relevant to the information structure of an online marketplace while at the same time, keeping the analysis tractable, the model is certainly not without limitations.

First, we assumed that each seller who participates in the platform only has one unit of the product in stock. In reality, when each seller represents a small business, they can carry multiple products in the inventory. However, we can extend our model without sacrificing any results to include the case where each seller has n units of the product for sale.

Second, products from different sellers in our model are perfectly substitutable as we focus on customers' buying and selling decisions of a particular product. In comparison, Chen et al. (2016) consider products that are not at all substitutable. Future work is needed to shed more light on the intermediate case where products from different sellers are substitutable to a certain extent.

Third, we modeled information revelation as discrete zero-one decisions in our framework for the sake of simplicity. Although market information is quite naturally an all-or-nothing decision, product information may have different levels that can be shared with the customers. Even so, the all-or-nothing product information revelation decision is well justified by the literature, see Lewis and Sappington (1991, 1994) and Johnson and Myatt (2006).

Fourth, we have largely derived our analytical results based on uniform distributions, under which the analyses were already complicated and in some cases intractable (e.g., the analytical characterization of T^P was not feasible). Yet, we have performed extensive numerical experiments based on the more general Beta distribution, and the key results and insights of the paper all continue to hold qualitatively.

Fifth, we studied the benefits of information which is only half of the story—the costs of information need to be assessed to determine the optimal strategy for the platform. In fact, some of the online marketplaces do not currently provide product and/or market information to customers due to the concerns about the cost of implementing the relevant platform technologies. Our work suggests that this may not be an adverse outcome under certain situations (e.g., sales volume maximization under a buyer's market).

Finally, we did not consider the platform's optimal commission problem, which is an interesting research question to pursue that is beyond the scope of the current paper. We refer interested readers to Hu and Zhou (2017) which studies the optimal commission structure of an on-demand matching platform. Similarly, we considered a monopolist platform in this paper as is typically done in the literature (see, e.g., Cachon et al. (2017), Taylor (2018) and Benjaafar et al. (2018)). Under oligopoly settings, how competition influences the online marketplaces' information decisions can be an interesting extension to this research, although analytical tractability can be in doubt.

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Electronic Companion to the "The Role of Product and Market Information in an Online Marketplace" paper

Proof of Theorem 1 Recall that the goal of each seller is to determine the listing price p, in order to maximize her individual expected revenue given in Eq. (1). Let $P^N(x)$ be an equilibrium selling probability under the sellers' game and consider the optimal selling price decision for the seller with a zero reservation value. Denote A as the maximizer set for $g(x) \doteq x \cdot P^N(x)$ (i.e., $A \doteq \arg\max_{x\in[0,1]} \{g(x)\}$). We will first show that A is a single non-empty closed interval, denoted by $[\underline{p}, \tilde{p}]$. The non-empty property of A directly comes from the fact that $P^N(x)$ is a non-increasing and left-contiunous function.

Next, we will show that A includes only one interval by *reductio ad absurdum*. Denote g^* as the maximum value for $g(x)$. If there are two adjacent intervals A_1 and A_2 in A (i.e., $A_1 \doteq [a_1, b_1] \subset A$, $A_2 \doteq [a_2, b_2] \subset A$, and $A_1 \cap A_2 = \varnothing$), then we denote the inteval between A_1 and A_2 as $B \doteq (b_1, a_2)$. By construction, no seller will charge a price in interval B. Consider the seller with a zero reservation value. Her surplus at an arbitrary point $b \in B$ can be expressed as $g(b) = b \cdot Pr\left\{D_b(1-b) > D_s \tilde{F}_s^N(b_1)\right\}$ where $\tilde{F}_s^N(b_1) \leq b_1$. Now, consider two scenarios: (i) If $\tilde{F}_s^N(b_1) > (1-b)$, then $g(b) = b \cdot \int_0^{(1-b)/\tilde{F}_s^N(b_1)} \int_{D_s \cdot \tilde{F}_s^N(b_1)/(1-b)}^1 dD_b dD_s = \frac{1}{2\tilde{F}_s^N(b_1)}\left(-\left(b-\frac{1}{2}\right)^2 + \frac{1}{4}\right)$, which is a strictly concave function; note that $g(b_1) = g(a_2) = g^* > g(b)$, which contradicts the fact that $g(b)$ is a strictly concave function; (ii) If $\tilde{F}_s^N(b_1) \leq (1-b)$, then $g(b) = b \cdot \int_0^1 \int_{D_s \cdot \tilde{F}_s^N(b_1)/(1-b)}^1 dD_b dD_s = b\left(1 - \frac{1}{2}\frac{c}{(1-b)}\right)$, whose first-order derivative is $\frac{1}{2(1-b)^2}\left(2(b+1)^2 - \tilde{F}_s^N(b_1)\right) > 0$, which contradicts the fact that $g(b_1) = g(a_2) = g^* > g(b)$. Therefore, there is only one interval in A.

To show that A is a closed interval, we need to rule out two cases: (i) If there is a jump at $\underline{p}$. As $g(x)$ is maximized at A, $g(x)$ must jump upwards at $\underline{p}$, which is not possible given that $P^N(x)$ is a non-increasing function and x is a continuous function. (ii) If there is a downward jump at $\tilde{p}$ and $\lim_{x\to\tilde{p}^-} g(x) > g(\tilde{p})$, which is also not possible given that $P^N(x)$ is a left–continuous function and x is a continuous function.

Therefore, we can define

$$\tilde{p} = \max\{\arg\max_p \{p \cdot P^N(p)\}\}, \tag{EC.1}$$

and correspondingly present the CDF of the sellers' equilibrium listing price distribution as

$$\tilde{F}_s^N(p) = \begin{cases} \underline{F}_s^N(p) & \text{if } 0 \leq p \leq \tilde{p}; \\ F_s(p) & \text{if } \tilde{p} \leq p \leq 1. \end{cases} \tag{EC.2}$$

If a seller's listing price is $p < \tilde{p}$, and her corresponding expected revenue satisfies $p \cdot P^N(p) < \tilde{p} \cdot P^N(\tilde{p})$, she always have an incentive to increase her price to $\tilde{p}$. In other words, the equilibrium price p of any seller whose reservation value $v_s < \tilde{p}$ must satisfy $p \cdot P^N(p) = \tilde{p} \cdot P^N(\tilde{p})$ in equilibrium.

Define $\underline{p} = \tilde{p} \cdot P^N(\tilde{p})$, we can deduce that $P^N(\underline{p}) = 1$. When a seller's listing price is $p < \underline{p}$, the corresponding revenue $p \cdot P^N(p) \leq p < \underline{p} = \tilde{p} \cdot P^N(\tilde{p})$, so the seller always has an incentive to increase her listing price to $\underline{p}$. That is, no seller would choose a price $p < \underline{p}$ in equilibrium. Thus, at any price $p < \underline{p}$, the corresponding selling probability of the product is $P^N(p) = 1$. Therefore, we have

$$p \cdot P^N(p) = \begin{cases} p \cdot 1 & \text{if } 0 \leq p \leq \underline{p}; \\ \tilde{p} \cdot P^N(\tilde{p}) & \text{if } \underline{p} \leq p \leq \tilde{p}. \end{cases} \tag{EC.3}$$

Depending on the realized values of D_b and D_s, a seller will be able to sell the product at price p if and only if $D_b \geq D_s \cdot \tilde{F}_s^N(p) \Leftrightarrow D_s \leq \frac{D_b}{\tilde{F}_s^N(p)}$, when $p \leq 0.5$. It follows from Eq. (2) that, when $p \leq 0.5$, we have

$$P^N(p) = E_{D_s,D_b}\left[\mathbb{1}\left\{D_b \geq D_s \cdot \tilde{F}_s^N(p)\right\}\right] = \int_0^s \int_{D_s \cdot \tilde{F}_s^N(p)}^b \frac{1}{bs} dD_b dD_s = \int_0^b \int_0^{\frac{D_b}{\tilde{F}_s^N(p)}} \frac{1}{bs} dD_s dD_b \quad \text{(EC.4)}$$

On the other hand, when $0.5 < p \leq 1$, it is clear from Eq. (2) that $P^N(p) = 0$. From Eq. (EC.1), we know $\tilde{p} \leq 0.5$.

Next we will consider two big cases, namely, $b \leq s$ and $b \geq s$. Specifically, when $b \leq s$, let us use θ to denote the solution of p to $b = s \cdot \tilde{F}_s^N(p)$, and we will discuss two subcases of $0 \leq \theta \leq 0.5$ and $0.5 \leq \theta \leq 1$. When $0 \leq \theta \leq 0.5$, we will need to consider two more scenarios $\tilde{p} \leq \theta$ and $\tilde{p} \geq \theta$. When $0.5 \leq \theta \leq 1$, we will only need to consider the case $\tilde{p} \leq \theta$ since $\tilde{p} \leq 0.5$. In other words, we will discuss the following four cases altogether, to find any possible solution of $\tilde{p}$ to Eq. (EC.1): Case (1) $b \leq s$ and $\tilde{p} \leq \theta \leq 0.5$; Case (2) $b \leq s$ and $\theta \leq \tilde{p} \leq 0.5$; Case (3) $b \leq s$ and $\tilde{p} \leq 0.5 \leq \theta$; and Case (4) $b \geq s$.

Case (1) When $b \leq s$ and $\tilde{p} \leq \theta \leq 0.5$. Case (1a). When $0 \leq p \leq \tilde{p}(\leq \theta)$, we have $s \cdot \tilde{F}_s^N(p) \leq b$. Also, we have $\tilde{F}_s^N(p) = \underline{F}_s^N(p)$ from Eq. (EC.2). It follows from Eqn. (EC.3) that $P^N(p) = \int_0^s \int_{D_s \underline{F}_s^N(p)}^b \frac{1}{bs} dD_b dD_s = 1 - \frac{s}{2b}\underline{F}_s^N(p)$. Case (1b). When $\tilde{p} \leq p \leq \theta$, we have $s \cdot \tilde{F}_s^N(p) \leq b$ and $\tilde{F}_s^N(p) = F_s(p)$. It follows from Eqn. (EC.3) that $P^N(p) = \int_0^s \int_{D_s F_s(p)}^b \frac{1}{bs} dD_b dD_s = 1 - \frac{s}{2b}p$. Case (1c). When $(\tilde{p} \leq)\theta \leq p \leq 0.5$, we also have $\tilde{F}_s^N(p) = F_s(p)$ from Eqn. (EC.2). However, because $s \cdot \tilde{F}_s^N(p) \geq b$, it is easier to calculate $P^N(p)$ using Eqn. (EC.4). It follows that $P^N(p) = \int_0^b \int_0^{\frac{D_b}{F_s(p)}} \frac{1}{bs} dD_s dD_b = \frac{b}{2s}\frac{1}{p}$. Case (1d). When $0.5 < p \leq 1$, we have $P^N(p) = 0$. Combining Cases (1a)-(1d), we have

$$p \cdot P^N(p) = \begin{cases} p\left[1 - \frac{s}{2b}\underline{F}_s^N(p)\right] & \text{if } 0 \leq p \leq \tilde{p}; \\ p\left[1 - \frac{s}{2b}p\right] & \text{if } \tilde{p} \leq p \leq \theta; \\ p\left[\frac{b}{2s}\frac{1}{p}\right] & \text{if } \theta \leq p \leq 0.5; \\ 0 & \text{if } 0.5 < p \leq 1. \end{cases} \quad \text{(EC.5)}$$

By comparing Eqn. (EC.5) and Eqn. (EC.3), $p \cdot P^N(p)$ must have the following structure

$$p \cdot P^N(p) = \begin{cases} p\left[1 - \frac{s}{2b}\underline{F}_s^N(p)\right] = p & \text{if } 0 \leq p \leq \underline{p}; \\ p\left[1 - \frac{s}{2b}\underline{F}_s^N(p)\right] = \tilde{p} \cdot P^N(\tilde{p}) & \text{if } \underline{p} \leq p \leq \tilde{p}; \\ p\left[1 - \frac{s}{2b}p\right] & \text{if } \tilde{p} \leq p \leq \theta; \\ p\left[\frac{b}{2s}\frac{1}{p}\right] & \text{if } \theta \leq p \leq 0.5; \\ 0 & \text{if } 0.5 < p \leq 1. \end{cases} \quad \text{(EC.6)}$$

Eqn. (EC.6), together with Eqn. (EC.2) and the fact that no customer charges more than $\tilde{p}$ unless the reservation price constraint is binding, we have

$$\tilde{F}_s^N(p) = \begin{cases} \underline{F}_s^N(p) = 0 & \text{if } 0 \leq p \leq \underline{p}; \\ \underline{F}_s^N(p) = \frac{2b}{s}\left(1 - \frac{\tilde{p} \cdot P^N(\tilde{p})}{p}\right) & \text{if } \underline{p} \leq p \leq \tilde{p}; \\ F_s(p) = p & \text{if } \tilde{p} \leq p \leq 1. \end{cases} \quad \text{(EC.7)}$$

From Eqn. (EC.6), we know that, when $\tilde{p} \leq p \leq \theta$, $p \cdot P^N(p) = p\left[1 - \frac{s}{2b}p\right] = p - \frac{s}{2b}p^2$ is concave in p by first increasing up to the point $p^* = \frac{b}{s}$ and then decreasing (the maximum value obtained at p^* is $p^* \cdot P(p^*) = \frac{b}{2s}$); when $\theta \leq p \leq 0.5$, $p \cdot P^N(p) = p\left[\frac{b}{2s}\frac{1}{p}\right] = \frac{b}{2s}$. Furthermore, we have $b = s \cdot \tilde{F}_s^N(\theta)$ by definition, and $\tilde{F}_s^N(\theta) = \theta$ from Eqn. (EC.7), we can derive $\theta = \frac{b}{s}$ and easily verify that $p \cdot P^N(p)$ is continuous at $\theta = \frac{b}{s}$. To ensure that the maximizers of $p\cdot P^N(p)$ are indeed $[\underline{p},\tilde{p}]$, we need to have $p^*\le\tilde{p}$. Therefore, we can deduce that the candidate $\tilde{p}$ satisfies $p^*\le\tilde{p}\le\theta$, or equivalently, $\tilde{p}=\frac{b}{s}$. Moreover, from $\theta\le 0.5$, we have the precondition $b\le\frac{s}{2}$.

Case (2) When $b\le s$, and $\theta\le\tilde{p}\le 0.5$. Case (2a). When $0\le p\le\theta(\le\tilde{p})$, we have $\tilde{F}_s^N(p)=\underline{F}_s^N(p)$ and $s\cdot\tilde{F}_s^N(p)\le b$. Thus, $P^N(p)$ has the same expression as Case (1a). Case (2b). When $\theta\le p\le\tilde{p}(\le 0.5)$, we have $s\cdot\tilde{F}_s^N(p)\ge b$ and $\tilde{F}_s^N(p)=\underline{F}_s^N(p)$, it follows that $P^N(p)=\int_0^b\int_0^{\frac{D_b}{\underline{F}_s^N(p)}}\frac{1}{bs}dD_s dD_b=\frac{b}{2s}\frac{1}{\underline{F}_s^N(p)}$. Case (2c). When $(\theta\le)\tilde{p}\le p\le 0.5$, we have $s\cdot\tilde{F}_s^N(p)\ge b$ and $\tilde{F}_s^N(p)=F_s(p)$. It follows that $P^N(p)$ has the same expression as in Case (1c). Case (2d). When $0.5<p\le 1$, we have $P^N(p)=0$. Combining Cases (2a)-(2d), we have

$$p\cdot P^N(p)=\begin{cases}p\left[1-\frac{s}{2b}\underline{F}_s^N(p)\right] & \text{if } 0\le p\le\theta;\\ p\left[\frac{b}{2s}\frac{1}{\underline{F}_s^N(p)}\right] & \text{if } \theta\le p\le\tilde{p};\\ p\left[\frac{b}{2s}\frac{1}{p}\right] & \text{if } \tilde{p}\le p\le 0.5;\\ 0 & \text{if } 0.5<p\le 1.\end{cases}\tag{EC.8}$$

By comparing Eqn. (EC.8) and Eqn. (EC.3), we have $p\cdot P^N(p)$ for two subcases when $\underline{p}\le\theta$ and when $\underline{p}\ge\theta$.

On one hand, when $\underline{p}\le\theta(\le\tilde{p}\le 0.5)$, we have

$$p\cdot P^N(p)=\begin{cases}p\left[1-\frac{s}{2b}\underline{F}_s^N(p)\right]=p & \text{if } 0\le p\le\underline{p};\\ p\left[1-\frac{s}{2b}\underline{F}_s^N(p)\right]=\tilde{p}\cdot P^N(\tilde{p}) & \text{if } \underline{p}\le p\le\theta;\\ p\left[\frac{b}{2s}\frac{1}{\underline{F}_s^N(p)}\right]=\tilde{p}\cdot P^N(\tilde{p}) & \text{if } \theta\le p\le\tilde{p};\\ p\left[\frac{b}{2s}\frac{1}{p}\right] & \text{if } \tilde{p}\le p\le 0.5;\\ 0 & \text{if } 0.5<p\le 1.\end{cases}\tag{EC.9}$$

Eqn. (EC.9), together with Eqn. (EC.2) and the fact that no customer charges more than $\tilde{p}$ unless the reservation price constraint is binding, we have

$$\tilde{F}_s^N(p)=\begin{cases}0 & \text{if } 0\le p\le\underline{p};\\ \frac{2b}{s}\left(1-\frac{\tilde{p}\cdot P^N(\tilde{p})}{p}\right) & \text{if } \underline{p}\le p\le\theta;\\ \frac{b}{2s}\frac{p}{\tilde{p}\cdot P^N(\tilde{p})} & \text{if } \theta\le p\le\tilde{p};\\ p & \text{if } \tilde{p}\le p\le 1.\end{cases}\tag{EC.10}$$

Because $\tilde{F}_s^N(p)$ needs to be continuous, we know that $\tilde{F}_s^N(\theta^-)=\tilde{F}_s^N(\theta^+)\Leftrightarrow\frac{2b}{s}\left(1-\frac{\tilde{p}\cdot P^N(\tilde{p})}{\theta}\right)=\frac{b}{2s}\frac{\theta}{\tilde{p}\cdot P^N(\tilde{p})}\Leftrightarrow$ $\theta=2\tilde{p}\cdot P^N(\tilde{p})$. That is, $\tilde{F}_s^N(\theta)=\tilde{F}_s^N(\theta^-)=\tilde{F}_s^N(\theta^+)=\frac{b}{s}$ and it is easy to verify that $b=s\cdot\tilde{F}_s^N(\theta)$.

On the other hand, when $\underline{p}\ge\theta$, we have from Eqn. (EC.8) and Eqn. (EC.3) that

$$p\cdot P^N(p)=\begin{cases}p\left[1-\frac{s}{2b}\underline{F}_s^N(p)\right]=p & \text{if } 0\le p\le\theta;\\ p\left[\frac{b}{2s}\frac{1}{\underline{F}_s^N(p)}\right]=p & \text{if } \theta\le p\le\underline{p};\\ p\left[\frac{b}{2s}\frac{1}{\underline{F}_s^N(p)}\right]=\tilde{p}\cdot P^N(\tilde{p}) & \text{if } \underline{p}\le p\le\tilde{p};\\ p\left[\frac{b}{2s}\frac{1}{p}\right] & \text{if } \tilde{p}\le p\le 0.5;\\ 0 & \text{if } 0.5<p\le 1.\end{cases}\tag{EC.11}$$

And it follows that

$$\tilde{F}_s^N(p)=\begin{cases}0 & \text{if } 0\le p\le\theta;\\ \frac{b}{2s} & \text{if } \theta\le p\le\underline{p};\\ \frac{b}{2s}\frac{p}{\tilde{p}\cdot P^N(\tilde{p})} & \text{if } \underline{p}\le p\le\tilde{p};\\ p & \text{if } \tilde{p}\le p\le 1.\end{cases}\tag{EC.12}$$

However, in this subcase, $\tilde{F}_s^N(p)$ is not continuous at $p=\theta$, so there cannot be an equilibrium when $\underline{p}>\theta$. So for Case 2, the candidate $\tilde{p}$ satisfies $(\underline{p}\le)\theta\le\tilde{p}\le 0.5$.

To summarize all the analysis so far, from Case (1), we know that if $\tilde{p}=\frac{b}{s}$, $p\cdot P^N(p)$ is given by Eqn. (EC.6), and $\tilde{F}_s^N(p)$ is given by Eqn. (EC.7). Therefore, there is a unique equilibrium in Case (1). From Case (2), we know that if $\tilde{p}\ge\theta(=\frac{b}{s})$, $p\cdot P^N(p)$ is given by Eqn. (EC.9), and $\tilde{F}_s^N(p)$ is given by Eqn. (EC.10), also, $\tilde{p}\cdot P^N(\tilde{p})$ decreases in $\tilde{p}$. As every $\tilde{p}$ in the set $\left[\frac{b}{s},1\right]$ is a equilibrium in the seller's game, we will identify the Pareto-dominant equilibrium in which sellers' expected revenue is maximized. In particular, combining Case (1) and Case (2), we know that in the case of $\theta\le 0.5$ (i.e., $b\le s/2$), when $\tilde{p}=\frac{b}{s}$, $\tilde{p}\cdot P^N(\tilde{p})$ achieves its maximum. Together with Eqn. (EC.1), we know that when $\tilde{p}=\frac{b}{s}$, all the sellers' expected revenue $p\cdot P^N(p)$ also achieve the maximum. That is, $\tilde{p}=\frac{b}{s}$ is the Pareto-dominant solution, with corresponding $\tilde{p}\cdot P^N(\tilde{p})=\frac{b}{2s}$, in the case of $b\le s/2$. Also, $p\cdot P^N(p)$ is given by Eqn. (EC.6), and $\tilde{F}_s^N(p)$ is given by Eqn. (EC.7).

Case (3). When $b\le s$ and $\tilde{p}\le 0.5\le\theta$. Case (3a). When $0\le p\le\tilde{p}(\le 0.5\le\theta)$, we have $s\cdot\tilde{F}_s^N(p)\le b$. Also, we have $\tilde{F}_s^N(p)=\underline{F}_s^N(p)$ from Eqn. (EC.2). It follows from Eqn. (EC.3) that $P^N(p)=\int_0^s\int_{D_s\underline{F}_s^N(p)}^b\frac{1}{bs}dD_b dD_s=1-\frac{s}{2b}\underline{F}_s^N(p)$. Case (3b). When $\tilde{p}\le p\le 0.5(\le\theta)$, we have $s\cdot\tilde{F}_s^N(p)\le b$ and $\tilde{F}_s^N(p)=F_s(p)$. It follows from Eqn. (EC.3) that $P^N(p)=\int_0^s\int_{D_sF_s^N(p)}^b\frac{1}{bs}dD_b dD_s=1-\frac{s}{2b}p$. Case (3c). When $0.5<p\le 1$, we have $P^N(p)=0$. Combining Cases (3a)-(3c), we have

$$p\cdot P^N(p)=\begin{cases}p\left[1-\frac{s}{2b}\underline{F}_s^N(p)\right] & \text{if } 0\le p\le\tilde{p};\\ p\left[1-\frac{s}{2b}p\right] & \text{if } \tilde{p}\le p\le 0.5;\\ 0 & \text{if } 0.5<p\le 1.\end{cases}\tag{EC.13}$$

By comparing Eqn. (EC.13) and Eqn. (EC.3), $p\cdot P^N(p)$ must have the following structure

$$p\cdot P^N(p)=\begin{cases}p\left[1-\frac{s}{2b}\underline{F}_s^N(p)\right]=p & \text{if } 0\le p\le\underline{p};\\ p\left[1-\frac{s}{2b}\underline{F}_s^N(p)\right]=\tilde{p}\cdot P^N(\tilde{p}) & \text{if } \underline{p}\le p\le\tilde{p};\\ p\left[1-\frac{s}{2b}p\right] & \text{if } \tilde{p}\le p\le 0.5;\\ 0 & \text{if } 0.5<p\le 1.\end{cases}\tag{EC.14}$$

Eqn. (EC.14), together with Eqn. (EC.2) and the fact that no customer charges more than $\tilde{p}$ unless the reservation price constraint is binding, we have

$$\tilde{F}_s^N(p)=\begin{cases}\underline{F}_s^N(p)=0 & \text{if } 0\le p\le\underline{p};\\ \underline{F}_s^N(p)=\frac{2b}{s}\left(1-\frac{\tilde{p}\cdot P^N(\tilde{p})}{p}\right) & \text{if } \underline{p}\le p\le\tilde{p};\\ F_s(p)=p & \text{if } \tilde{p}\le p\le 1.\end{cases}\tag{EC.15}$$

From Eqn. (EC.14), we know that, when $\tilde{p}\le p\le 0.5$, $p\cdot P^N(p)=p\left[1-\frac{s}{2b}p\right]=p-\frac{s}{2b}p^2$ is concave in p by first increasing up to the point $p^*=\frac{b}{s}$ (if $\frac{b}{s}\le 0.5$) and then decreasing (the maximum value obtained at p^* is $p^*\cdot P(p^*)=\frac{b}{2s}$), or $p\cdot P^N(p)=p\left[1-\frac{s}{2b}p\right]=p-\frac{s}{2b}p^2$ is concave increasing in p to the point $p^*=0.5$ (if $\frac{b}{s}\ge 0.5$); when $0.5<p\le 1$, $p\cdot P^N(p)=0$. Furthermore, we have $b=s\cdot\tilde{F}_s^N(\theta)$ by definition, and $\tilde{F}_s^N(\theta)=\theta$ from Eqn. (EC.15) (i.e., $\tilde{p}\le 0.5\le\theta$), we can derive $\theta=\frac{b}{s}$ and easily verify that $p\cdot P^N(p)=0$ at $\theta=\frac{b}{s}(\ge 0.5)$. Therefore, we have $p^*=0.5$. To ensure that the maximizers of $p\cdot P^N(p)$ are indeed $[\underline{p},\tilde{p}]$, we need to have $p^*\le\tilde{p}$. Therefore, we can deduce that the candidate $\tilde{p}$ satisfies $p^*\le\tilde{p}\le 0.5$, or equivalently, $\tilde{p}=0.5$, and the corresponding $\tilde{p}\cdot P^N(\tilde{p})=\frac{1}{2}-\frac{s}{8b}$. Moreover, from $0.5\le\theta\le 1$, we have the precondition $\frac{s}{2}\le b\le s$.

Case (4) When $b\ge s$, we have $b\ge s\cdot\tilde{F}_s^N(p)$ for any p. Case (4a): when $0\le p\le\tilde{p}(\le 0.5)$, it is easier to calculate $P^N(p)$ using Eqn. (EC.3). Also, $\tilde{F}_s^N(p)=\underline{F}_s^N(p)$ from Eqn. (EC.2). It follows that $P^N(p)=$

$\int_0^s \int_{D_s \underline{F}_s^N(p)}^b \frac{1}{bs} dD_b dD_s = 1 - \frac{s}{2b}\underline{F}_s^N(p)$. Case (4b): When $\tilde{p} \le p \le 0.5$, we have $s \cdot \tilde{F}_s^N(p) \le b$ and $\tilde{F}_s^N(p) = F_s(p)$. It follows from Eqn. (EC.3) that $P^N(p) = \int_0^s \int_{D_s F_s^N(p)}^b \frac{1}{bs} dD_b dD_s = 1 - \frac{s}{2b}p$. Case (4c). When $0.5 < p \le 1$, we have $P^N(p) = 0$.

Combining Cases (4a)-(4c), we have

$$p \cdot P^N(p) = \begin{cases} p\left[1 - \frac{s}{2b}\underline{F}_s^N(p)\right] & \text{if } 0 \le p \le \tilde{p}; \\ p\left[1 - \frac{s}{2b}p\right] & \text{if } \tilde{p} \le p \le 0.5; \\ 0 & \text{if } 0.5 < p \le 1. \end{cases} \tag{EC.16}$$

By comparing Eqn. (EC.16) and Eqn. (EC.3), $p \cdot P^N(p)$ must have the following structure

$$p \cdot P^N(p) = \begin{cases} p\left[1 - \frac{s}{2b}\underline{F}_s^N(p)\right] = p & \text{if } 0 \le p \le \underline{p}; \\ p\left[1 - \frac{s}{2b}\underline{F}_s^N(p)\right] = \tilde{p} \cdot P^N(\tilde{p}) & \text{if } \underline{p} \le p \le \tilde{p}; \\ p\left[1 - \frac{s}{2b}p\right] & \text{if } \tilde{p} \le p \le 0.5; \\ 0 & \text{if } 0.5 < p \le 1. \end{cases} \tag{EC.17}$$

Eqn. (EC.17), together with Eqn. (EC.2) and the fact that no customer charges more than $\tilde{p}$ unless the reservation price constraint is binding, we have

$$\tilde{F}_s^N(p) = \begin{cases} \underline{F}_s^N(p) = 0 & \text{if } 0 \le p \le \underline{p}; \\ \underline{F}_s^N(p) = \frac{2b}{s}\left(1 - \frac{\tilde{p} \cdot P^N(\tilde{p})}{p}\right) & \text{if } \underline{p} \le p \le \tilde{p}(\le 0.5); \\ F_s(p) = p & \text{if } \tilde{p} \le p \le 1. \end{cases} \tag{EC.18}$$

It is also easy to verify that when $\tilde{p} \le p \le 0.5$, $p \cdot P^N(p) = p\left[1 - \frac{s}{2b}p\right] = p - \frac{s}{2b}p^2$ is concave increasing in p (since the extreme point $\frac{b}{s} \ge 1$), then the maximize $p^* = 0.5$. To ensure that the maximizers of $p \cdot P^N(p)$ are indeed $[\underline{p}, \tilde{p}]$, we need to have $p^* \le \tilde{p}$. Therefore, we can deduce that the candidate $\tilde{p}$ satisfies $p^* \le \tilde{p} \le 0.5$, or equivalently, $\tilde{p} = 0.5$, and the corresponding $\tilde{p} \cdot P^N(\tilde{p}) = \frac{1}{2} - \frac{s}{8b}$.

Combining Case (3) and Case (4), we know that in the case of $b \ge s/2$, when $\tilde{p} = 0.5$, $\tilde{p} \cdot P^N(\tilde{p})$ achieves its maximum. Together with Eqn. (EC.1), we know that when $\tilde{p} = 0.5$, all the sellers' expected revenue $p \cdot P^N(p)$ also achieve the maximum. That is, $\tilde{p} = 0.5$ is the Pareto-dominant solution, with corresponding $\tilde{p} \cdot P^N(\tilde{p}) = \frac{1}{2} - \frac{s}{8b}$. Also, $p \cdot P^N(p)$ is given by Eqn. (EC.14) (or Eqn. (EC.17)), and $\tilde{F}_s^N(p)$ is given by Eqn. (EC.15) (or Eqn. (EC.18)).

Finally, we will establish that when $b \le s/2$ (resp. $b \ge s/2$), $\tilde{p} = \frac{b}{s}$ (resp. $\tilde{p} = 0.5$) is indeed a Pareto-dominant equilibrium solution. In particular, we need to show that they satisfy $\int_0^1 \tilde{f}_s^N(p)dp = 1$ and $\tilde{F}_s^N(p) \le F_s(p)$

When $b \le s/2$, from Eqn. (EC.7), we can deduce the probability density function, $\tilde{f}_s^N(p)$, as follows:

$$\tilde{f}_s^N(p) = \begin{cases} 0 & \text{if } 0 \le p \le \underline{p}; \\ \frac{2b}{s}\frac{\tilde{p} \cdot P^N(\tilde{p})}{p^2} & \text{if } \underline{p} \le p \le \tilde{p}; \\ 1 & \text{if } \tilde{p} \le p \le 1. \end{cases} \tag{EC.19}$$

Because $\tilde{p} \le \theta$, we have $P^N(\tilde{p}) = 1 - \frac{s}{2b}\tilde{p}$. Thus, $\int_0^1 \tilde{f}_s^N(p)dp = \int_{\underline{p} = \tilde{p} \cdot P^N(\tilde{p})}^{\tilde{p}} \frac{2b}{s}\frac{\tilde{p} \cdot P^N(\tilde{p})}{p^2}dp + \int_{\tilde{p}}^1 1dp = \frac{2b}{s}(1 - P^N(\tilde{p})) + (1 - \tilde{p}) = 1$. In addition, when $\tilde{p} = \frac{b}{s}$, we show in what follows that, for any p, we have $\tilde{F}_s^N(p) \le F_s(p)$:

From Eqn. (EC.7), we know that: 1. If $p \le \underline{p}$, we have $\tilde{F}_s^N(p) = 0 < p = F_s(p)$; 2. If $\underline{p} \le p \le \tilde{p}$, we have $\tilde{F}_s^N(p) - F_s(p) = \frac{2b}{s}\left(1 - \frac{\tilde{p} \cdot P^N(\tilde{p})}{p}\right) - p$, where $\tilde{p} \cdot P^N(\tilde{p}) = \frac{b}{2s}$. Since the second-order derivative $\frac{\partial^2(\tilde{F}_s^N(p) - F_s(p))}{\partial p^2} = -\frac{2b^2}{p^3 s^2} \le 0$, then the first-order derivative $\frac{\partial(\tilde{F}_s^N(p) - F_s(p))}{\partial p} = \frac{b^2}{p^2 s^2} - 1$ is decreasing in p, that is $\frac{\partial(\tilde{F}_s^N(p) - F_s(p))}{\partial p} \ge$

$\frac{\partial(\tilde{F}_s^N(p)-F_s(p))}{\partial p}|_{p=\tilde{p}}=0$. Therefore, we can deduce that $\tilde{F}_s^N(p)-F_s(p)$ is increasing in p, then $\tilde{F}_s^N(p)-F_s(p)\le \tilde{F}_s^N(\tilde{p})-F_s(\tilde{p})=0$, i.e., $\tilde{F}_s^N(p)\le F_s(p)$. 3. If $p\ge\tilde{p}$, we have $\tilde{F}_s^N(p)=p=F_s(p)$.

When $b\ge s/2$, from Eqn. (EC.15), we can deduce the probability density function, $\tilde{f}_s^N(p)$, as follows:

$$\tilde{f}_s^N(p)=\begin{cases}0 & \text{if } 0\le p\le \underline{p};\\ \frac{2b}{s}\frac{\tilde{p}\cdot P^N(\tilde{p})}{p^2} & \text{if } \underline{p}\le p\le\tilde{p};\\ 1 & \text{if } \tilde{p}\le p\le 1.\end{cases} \tag{EC.20}$$

Because $\tilde{p}\le 0.5\le\theta$, we have $P^N(\tilde{p})=1-\frac{s}{2b}\tilde{p}$. Thus, $\int_0^1\tilde{f}_s^N(p)dp=\int_{\underline{p}=\tilde{p}\cdot P^N(\tilde{p})}^{\tilde{p}}\frac{2b}{s}\frac{\tilde{p}\cdot P^N(\tilde{p})}{p^2}dp+\int_{\tilde{p}}^1 1dp=\frac{2b}{s}(1-P^N(\tilde{p}))+(1-\tilde{p})=1$. In addition, when $\tilde{p}=0.5$, we show in what follows that, for any p, we have $\tilde{F}_s^N(p)\le F_s(p)$:

From Eqn. (EC.15), we know that: 1. If $p\le\underline{p}$, we have $\tilde{F}_s^N(p)=0<p=F_s(p)$; 2. If $\underline{p}\le p\le\tilde{p}$, we have $\tilde{F}_s^N(p)-F_s(p)=\frac{2b}{s}(1-\frac{\tilde{p}\cdot P^N(\tilde{p})}{p})-p$, where $\tilde{p}\cdot P^N(\tilde{p})=\frac{1}{2}-\frac{s}{8b}$. Since the second-order derivative $\frac{\partial^2(\tilde{F}_s^N(p)-F_s(p))}{\partial p^2}=\frac{s-4b}{2p^3s}\le 0$, then the first-order derivative $\frac{\partial(\tilde{F}_s^N(p)-F_s(p))}{\partial p}=\frac{\frac{b}{s}-\frac{1}{4}}{p^2}-1$ is decreasing in p, that is $\frac{\partial(\tilde{F}_s^N(p)-F_s(p))}{\partial p}\ge\frac{\partial(\tilde{F}_s^N(p)-F_s(p))}{\partial p}|_{p=\tilde{p}}=0$. Therefore, we can deduce that $\tilde{F}_s^N(p)-F_s(p)$ is increasing in p, then $\tilde{F}_s^N(p)-F_s(p)\le\tilde{F}_s^N(\tilde{p})-F_s(\tilde{p})=0$, i.e., $\tilde{F}_s^N(p)\le F_s(p)$. 3. If $p\ge\tilde{p}$, we have $\tilde{F}_s^N(p)=p=F_s(p)$. □

Proof of Corollary 1 From Theorem 1, we know that when $b\le s/2$, we have $\frac{\partial(\frac{2b}{s}(1-\frac{b}{2s\cdot p}))}{\partial b}\le 0$, $\frac{\partial(\frac{b}{2s})}{\partial b}\ge 0$, and $\frac{\partial(\frac{b}{s})}{\partial b}\ge 0$. That is, $\tilde{F}_s^N(p)$ decreases in b. Similarly, we have $\frac{\partial(\frac{2b}{s}(1-\frac{b}{2s\cdot p}))}{\partial s}\ge 0$, $\frac{\partial(\frac{b}{2s})}{\partial s}\le 0$, and $\frac{\partial(\frac{b}{s})}{\partial s}\le 0$, which together suggest that $\tilde{F}_s^N(p)$ increases in s. On the other hand, when $b>s/2$, the result that $\tilde{F}_s^N(p)$ decreases in b directly comes from the facts that $\frac{\partial(\frac{2b}{s}(1-(0.5-\frac{s}{8b})\frac{1}{p}))}{\partial b}\le 0$ and $\frac{\partial(0.5-\frac{s}{8b})}{\partial b}\ge 0$; and the result that $\tilde{F}_s^N(p)$ increases in s comes from $\frac{\partial(\frac{2b}{s}(1-(0.5-\frac{s}{8b})\frac{1}{p}))}{\partial s}\ge 0$ and $\frac{\partial(0.5-\frac{s}{8b})}{\partial s}\le 0$. □

Proof of Proposition 1 We start with the first part of this proposition: Q^N increases in b and s. From Theorem 1, we can show that $q^N=d_b$ for $d_b/d_s\le 0.5$ and $q^N=d_s/2$ for $d_b/d_s>0.5$. Then, we can derive the expressions of Q^N for the following two cases:

$$Q^N=\begin{cases}\int\int_D q^N dD_b dD_s=\frac{b}{6s}(3s-2b) & \text{if } b/s\le 0.5;\\ \int\int_D q^N dD_b dD_s=\frac{s}{24b}(6b-s) & \text{if } b/s>0.5.\end{cases}$$

When $b/s\le 0.5$, we have $\frac{\partial Q^N}{\partial b}=\frac{1}{2}-\frac{2b}{3s}>0$, and $\frac{\partial Q^N}{\partial s}=\frac{b^2}{3s^2}>0$. Similarly, when $b/s>0.5$, we have $\frac{\partial Q^N}{\partial b}=\frac{s^2}{24b^2}>0$, and $\frac{\partial Q^N}{\partial s}=\frac{1}{4}-\frac{s}{12b}>0$.

Next, we consider the second part of this proposition: T^N increases in b and is unimodal in s. First, we need to derive the expression for T^N, and to this end, we consider two cases: $b\le s/2$ and $b>s/2$. For the first case where $b\le s/2$, recall that when $d_b\le d_s\cdot\tilde{F}_s^N(0.5)$, p_{max}^N is given by $d_b=d_s\cdot\tilde{F}_s^N(p_{max}^N)$; when $d_b>d_s\cdot\tilde{F}_s^N(0.5)$, $p_{max}^N=0.5$. Then, we have three possibilities: if $\frac{d_b}{d_s}\le\frac{b}{s}$, then we have $p_{max}^N\le\frac{b}{s}$, which indicates that p_{max}^N is given by $d_b=d_s\cdot\frac{2b}{s}(1-\frac{b}{2s}\frac{1}{p_{max}^N})$; if $\frac{b}{s}<\frac{d_b}{d_s}\le\frac{1}{2}$, then we have $\frac{b}{s}<p_{max}^N\le\frac{1}{2}$, which suggests that p_{max}^N is given by $d_b=d_s\cdot p_{max}^N$; and finally, if $\frac{d_b}{d_s}>\frac{1}{2}$, then we have $p_{max}^N=\frac{1}{2}$. Therefore, p_{max}^N is given by

$$p_{max}^N=\begin{cases}\frac{\frac{b}{2s}}{1-\frac{d_b}{d_s}\frac{s}{2b}} & \text{if } \frac{d_b}{d_s}\le\frac{b}{s};\\ \frac{d_b}{d_s} & \text{if } \frac{b}{s}<\frac{d_b}{d_s}\le\frac{1}{2};\\ \frac{1}{2} & \text{if } \frac{d_b}{d_s}>\frac{1}{2}.\end{cases}$$

Recall that when $b \le \frac{s}{2}$, we have

$$\tilde{f}_s^N(p) = \begin{cases} 0 & \text{if } p \le \frac{b}{2s}; \\ \frac{b^2}{s^2}\frac{1}{p^2} & \text{if } \frac{b}{2s} < p \le \frac{b}{s}; \\ 1 & \text{if } p > \frac{b}{s}. \end{cases} \text{ and } t^N = \begin{cases} d_s \cdot \int_0^{p^N_{max}} p \cdot \tilde{f}_s^N(p)\, dp & \text{if } \frac{d_b}{d_s} \le \frac{1}{2}; \\ d_s \cdot \int_0^{0.5} p \cdot \tilde{f}_s^N(p)\, dp & \text{if } \frac{d_b}{d_s} > \frac{1}{2}. \end{cases}$$

Then, we can explicitly express t^N as follows:

$$t^N = \begin{cases} d_s \cdot \int_{\frac{b}{2s}}^{\frac{\frac{b}{2s}}{1-\frac{d_b}{d_s}\frac{s}{2b}}} p \cdot \frac{b^2}{s^2}\frac{1}{p^2}\, dp = d_s \cdot \frac{b^2}{s^2}[\log(\frac{\frac{b}{2s}}{1-\frac{d_b}{d_s}\frac{s}{2b}}) - \log(\frac{b}{2s})] & \text{if } \frac{d_b}{d_s} \le \frac{b}{s}; \\ d_s \cdot \int_{\frac{b}{2s}}^{\frac{b}{s}} p \cdot \frac{b^2}{s^2}\frac{1}{p^2}\, dp + d_s \cdot \int_{\frac{b}{s}}^{\frac{d_b}{d_s}} p\, dp = d_s \cdot \frac{b^2}{s^2}[\log(\frac{b}{s}) - \log(\frac{b}{2s})] + d_s \cdot (\frac{1}{2}\frac{d_b^2}{d_s^2} - \frac{1}{2}\frac{b^2}{s^2}) & \text{if } \frac{b}{s} < \frac{d_b}{d_s} \le \frac{1}{2}; \\ d_s \cdot \int_{\frac{b}{2s}}^{\frac{b}{s}} p \cdot \frac{b^2}{s^2}\frac{1}{p^2}\, dp + d_s \cdot \int_{\frac{b}{s}}^{0.5} p\, dp = d_s \cdot \frac{b^2}{s^2}[\log(\frac{b}{s}) - \log(\frac{b}{2s})] + d_s \cdot (\frac{1}{8} - \frac{1}{2}\frac{b^2}{s^2}) & \text{if } \frac{d_b}{d_s} > \frac{1}{2}. \end{cases}$$

Therefore, when $b \le \frac{s}{2}$, we have $T^N = \int\int_D t^N dD_b dD_s = \frac{b^2(2-\log(4b)+\log(s))}{6s}$. Next, similarly when $b > \frac{s}{2}$, we can show that

$$p^N_{max} = \begin{cases} \frac{\frac{1}{2}-\frac{s}{8b}}{1-\frac{d_b}{d_s}\frac{s}{2b}} & \text{if } \frac{d_b}{d_s} \le \frac{1}{2}; \\ \frac{1}{2} & \text{if } \frac{d_b}{d_s} > \frac{1}{2}. \end{cases}$$

And, we have

$$\tilde{f}_s^N(p) = \begin{cases} 0 & \text{if } p \le \frac{1}{2} - \frac{s}{8b}; \\ (\frac{b}{s} - \frac{1}{4})\frac{1}{p^2} & \text{if } \frac{1}{2} - \frac{s}{8b} < p \le \tilde{p}; \\ 1 & \text{if } p > \tilde{p}. \end{cases} \text{ and } t^N = \begin{cases} d_s \int_0^{p^N_{max}} p\tilde{f}_s^N(p)dp & \text{if } \frac{d_b}{d_s} \le \frac{1}{2}; \\ d_s \int_0^{0.5} p\tilde{f}_s^N(p)dp & \text{if } \frac{d_b}{d_s} > \frac{1}{2}. \end{cases}$$

Then, t^N can be explicitly expressed as follows:

$$t^N = \begin{cases} d_s \cdot \int_{\frac{1}{2}-\frac{s}{8b}}^{\frac{\frac{1}{2}-\frac{s}{8b}}{1-\frac{d_b}{d_s}\frac{s}{2b}}} p \cdot (\frac{b}{s} - \frac{1}{4})\frac{1}{p^2}\, dp = d_s \cdot (\frac{b}{s} - \frac{1}{4})[\log(\frac{\frac{1}{2}-\frac{s}{8b}}{1-\frac{d_b}{d_s}\frac{s}{2b}}) - \log(\frac{1}{2} - \frac{s}{8b})] & \text{if } \frac{d_b}{d_s} \le \frac{1}{2}; \\ d_s \cdot \int_{\frac{1}{2}-\frac{s}{8b}}^{\frac{1}{2}} p \cdot (\frac{b}{s} - \frac{1}{4})\frac{1}{p^2}\, dp = d_s \cdot (\frac{b}{s} - \frac{1}{4})[\log(\frac{1}{2}) - \log(\frac{1}{2} - \frac{s}{8b})] & \text{if } \frac{d_b}{d_s} > \frac{1}{2}. \end{cases}$$

Therefore, when $b > \frac{s}{2}$, we have $T^N = \int\int_D t^N dD_b dD_s = \frac{(4b-s)\left(b\log\left(4-\frac{s}{b}\right)-b\log(4)+s\right)}{24b}$. To summarize, T^N is given by

$$T^N = \begin{cases} \frac{b^2(2-\log(4b)+\log(s))}{6s} & \text{if } \frac{b}{s} \le \frac{1}{2}; \\ \frac{(4b-s)\left(b\log\left(4-\frac{s}{b}\right)-b\log(4)+s\right)}{24b} & \text{if } \frac{b}{s} > \frac{1}{2}. \end{cases}$$

To show T^N increases in b, we start with the case of $\frac{b}{s} \le \frac{1}{2}$, where $\frac{\partial^3 T^N}{\partial b^3} = -\frac{1}{3bs} < 0$. That is, $\frac{\partial^2 T^N}{\partial b^2}$ decreases in b. Since as $b \to 0^+$, we have $\frac{\partial^2 T^N}{\partial b^2} = \frac{-2\log(4b)+2\log(s)+1}{6s} > 0$; when $b = s/2$, we have $\frac{\partial^2 T^N}{\partial b^2} = -\frac{\log(4)-1}{6s} < 0$. That is, $\frac{\partial T^N}{\partial b}$ first increases in b, then decreases in b. Since when $b \to 0^+$, we have $\frac{\partial T^N}{\partial b} = \frac{b(-2\log(4b)+2\log(s)+3)}{6s} > 0$; when $b = s/2$, we have $\frac{\partial T^N}{\partial b} = \frac{1}{4} - \frac{\log(2)}{6} > 0$. Therefore, T^N increases in b. In the case of $\frac{b}{s} > \frac{1}{2}$, we have $\frac{\partial T^N}{\partial b} = \frac{4b^2\log\left(4-\frac{s}{b}\right)-4b^2\log(4)+s(b+s)}{24b^2} > 0$, which suggests that T^N increases in b.

Finally, we will show T^N is unimodal in s. For the case of $\frac{b}{s} \le \frac{1}{2}$, we have $\frac{\partial T^N}{\partial s} = \frac{b^2(\log(b)-\log(s)-1+\log(4))}{6s^2} < 0$, which directly suggests that T^N decreases in s. As we have $T^N = \frac{1}{12}b(2-\log(2)) > 0$ for $s = 2b$ and we have $T^N = 0$ for $s \to +\infty$. In other words, when $\frac{b}{s} \le \frac{1}{2}$, we know that T^N decreases from $\frac{1}{12}b(2-\log(2))$ to 0. For the case of $\frac{b}{s} > \frac{1}{2}$, we have $\frac{\partial^2 T^N}{\partial s^2} = \frac{1}{24}\left(\frac{1}{4b-s} - \frac{2}{b}\right) < 0$. That is, $\frac{\partial T^N}{\partial s}$ decreases in s. As $\frac{\partial T^N}{\partial s} = \frac{-b\log\left(4-\frac{s}{b}\right)+b(3+\log(4))-2s}{24b} = \frac{1}{8} > 0$ for $s = 0$ and $\frac{\partial T^N}{\partial s} = \frac{1}{24}(\log(2)-1) < 0$ for $s = 2b$, we have that T^N first increases in s and then decreases in s. Since we have $T^N = 0$ for $s = 0$, we have $T^N = \frac{1}{12}b(2-\log(2))$ for $s = 2b$. That is, for the case of $\frac{b}{s} > \frac{1}{2}$, we know that T^N first increases from 0 to some maximum values and then decreases from the maximum value to $\frac{1}{12}b(2-\log(2))$. Combining these two cases, we know that T^N is continuous and is unimodal in s. ☐

Proof of Theorem 2 Similar to the proof for Theorem 1, we define the CDF of the sellers' equilibrium listing price distribution as follows.

$$\tilde{F}_s^P(p) = \begin{cases} \underline{F}_s^P(p) & \text{if } 0 \le p \le \tilde{p}; \\ F_s(p) & \text{if } \tilde{p} \le p \le 1 \end{cases} \tag{EC.21}$$

where

$$\tilde{p} = \max\{\arg\max_p \{p \cdot P^P(p)\}\}. \tag{EC.22}$$

We have

$$p \cdot P^P(p) = \begin{cases} p \cdot 1 & \text{if } 0 \le p \le \underline{p}; \\ \tilde{p} \cdot P^P(\tilde{p}) & \text{if } \underline{p} \le p \le \tilde{p}. \end{cases} \tag{EC.23}$$

Depending on the realized values of D_b and D_s, A seller will be able to sell the product at price p if and only if $D_b \cdot (1 - F_b(p)) \ge D_s \cdot \tilde{F}_s^P(p) \Leftrightarrow D_b \ge \frac{D_s \cdot \tilde{F}_s^P(p)}{1-F_b(p)} \Leftrightarrow D_s \le \frac{D_b(1-F_b(p))}{\tilde{F}_s^P(p)}$. It follows from Eqn. (3) that

$$\begin{aligned} P^P(p) &= \mathop{\mathbb{E}}_{D_s, D_b}\left[\mathbb{1}\left\{D_b \cdot (1 - F_b(p)) \ge D_s \cdot \tilde{F}_s^P(p)\right\}\right] \\ &= \int_0^s \int_{\frac{D_s \cdot \tilde{F}_s^P(p)}{1-F_b(p)}}^b \frac{1}{bs} dD_b dD_s \qquad \text{(EC.24)} \\ &= \int_0^b \int_0^{\frac{D_b(1-F_b(p))}{\tilde{F}_s^P(p)}} \frac{1}{bs} dD_s dD_b \qquad \text{(EC.25)} \end{aligned}$$

Let us use θ to denote the solution of p to $b \cdot (1 - F_b(p)) = s \cdot \tilde{F}_s^P(p)$. We will discuss two cases when $\tilde{p} \le \theta$ and when $\tilde{p} \ge \theta$, to find the possible solution of $\tilde{p}$ defined in Eqn. (EC.22).

Case (1). $\tilde{p} \le \theta$. Case (1a). When $0 \le p \le \tilde{p}(\le \theta)$, we have $\frac{s \cdot \tilde{F}_s^P(p)}{1-F_b(p)} \le b$. Also, we have $\tilde{F}_s^P(p) = \underline{F}_s^P(p)$ from Eqn. (EC.21). It follows from Eqn. (EC.24) that $P^P(p) = \int_0^s \int_{\frac{D_s \underline{F}_s^P(p)}{1-F_b(p)}}^b \frac{1}{bs} dD_b dD_s = 1 - \frac{s}{2b} \frac{\underline{F}_s^P(p)}{1-p}$. Case (1b). When $\tilde{p} \le p \le \theta$, we have $\frac{s \cdot \tilde{F}_s^P(p)}{1-F_b(p)} \le b$ and $\tilde{F}_s^P(p) = F_s(p)$. It follows from Eqn. (EC.24) that $P^P(p) = \int_0^s \int_{\frac{D_s F_s(p)}{1-F_b(p)}}^b \frac{1}{bs} dD_b dD_s = 1 - \frac{s}{2b} \frac{p}{1-p}$. Case (1c). When $(\tilde{p} \le)\theta \le p \le 1$, we also have $\tilde{F}_s^P(p) = F_s(p)$ from Eqn. (EC.21). However, because $\frac{s \cdot \tilde{F}_s^P(p)}{1-F_b(p)} \ge b$, it is easier to calculate $P^P(p)$ using Eqn. (EC.25). It follows that $P^P(p) = \int_0^b \int_0^{\frac{D_b(1-F_b(p))}{F_s(p)}} \frac{1}{bs} dD_s dD_b = \frac{b}{2s} \frac{1-p}{p}$.

Combining Cases (1a)-(1c), we have

$$p \cdot P^P(p) = \begin{cases} p\left[1 - \frac{s}{2b}\frac{\underline{F}_s^P(p)}{1-p}\right] & \text{if } 0 \le p \le \tilde{p}; \\ p\left[1 - \frac{s}{2b}\frac{p}{1-p}\right] & \text{if } \tilde{p} \le p \le \theta; \\ p\left[\frac{b}{2s}\frac{1-p}{p}\right] & \text{if } \theta \le p \le 1. \end{cases} \tag{EC.26}$$

By comparing Eqn. (EC.26) and Eqn. (EC.23), $p \cdot P^P(p)$ must have the following structure

$$p \cdot P^P(p) = \begin{cases} p\left[1 - \frac{s}{2b}\frac{\underline{F}_s^P(p)}{1-p}\right] = p & \text{if } 0 \le p \le \underline{p}; \\ p\left[1 - \frac{s}{2b}\frac{\underline{F}_s^P(p)}{1-p}\right] = \tilde{p} \cdot P^P(\tilde{p}) & \text{if } \underline{p} \le p \le \tilde{p}; \\ p\left[1 - \frac{s}{2b}\frac{p}{1-p}\right] & \text{if } \tilde{p} \le p \le \theta; \\ p\left[\frac{b}{2s}\frac{1-p}{p}\right] & \text{if } \theta \le p \le 1. \end{cases} \tag{EC.27}$$

Eqn. (EC.27), together with Eqn. (EC.21) and the fact that no customer charges more than $\tilde{p}$ unless the reservation price constraint is binding, we have

$$\tilde{F}_s^P(p) = \begin{cases} \underline{F}_s^P(p) = 0 & \text{if } 0 \le p \le \underline{p}; \\ \underline{F}_s^P(p) = \frac{2b}{s}(1-p)(1 - \frac{\tilde{p} \cdot P^P(\tilde{p})}{p}) & \text{if } \underline{p} \le p \le \tilde{p}; \\ F_s(p) = p & \text{if } \tilde{p} \le p \le 1. \end{cases} \tag{EC.28}$$

From Eqn. (EC.27), we know that, when $\tilde{p} \leq p \leq \theta$, $p \cdot P^P(p) = p\left[1 - \frac{s}{2b}\frac{p}{1-p}\right] = p - \frac{s}{2b}\frac{p^2}{1-p}$ is concave in p by first increasing up to the point $p^* = 1 - \sqrt{\frac{s}{2b+s}}$ and then decreasing (the maximum value obtained at p^* is $p^* \cdot P^P(p^*) = \frac{b+s-\sqrt{(2b+s)s}}{b}$); when $\theta \leq p \leq 1$, $p \cdot P^P(p) = p\left[\frac{b}{2s}\frac{1-p}{p}\right] = \frac{b}{2s}(1-p)$ is decreasing in p. Furthermore, we have $b \cdot (1 - F_b(\theta)) = s \cdot \tilde{F}_s^P(\theta)$ by definition, and $\tilde{F}_s^P(\theta) = \theta$ from Eqn. (EC.28), we can derive $\theta = \frac{b}{b+s}$ and easily verify that $p \cdot P^P(p)$ is continuous at $\theta = \frac{b}{b+s}$. To ensure that the maximizers of $p \cdot P^P(p)$ are indeed $[\underline{p}, \tilde{p}]$, we need to have $p^* \leq \tilde{p}$. Therefore, we can deduce that the candidate $\tilde{p}$ satisfies $p^* \leq \tilde{p} \leq \theta$, or equivalently, $1 - \sqrt{\frac{s}{2b+s}} \leq \tilde{p} \leq \frac{b}{b+s}$.

Case (2). $\tilde{p} \geq \theta$. Case (2a). When $0 \leq p \leq \theta (\leq \tilde{p})$, we have $\tilde{F}_s^P(p) = \underline{F}_s^P(p)$ and $\frac{s \cdot \tilde{F}_s^P(p)}{1-F_b(p)} \leq b$. Thus, $P^P(p)$ has the same expression as in Case 1a. Case (2b). When $\theta \leq p \leq \tilde{p}$, we have $\frac{s \cdot \tilde{F}_s^P(p)}{1-F_b(p)} \geq b$ and $\tilde{F}_s^P(p) = \underline{F}_s^P(p)$, it follows from Eqn. (EC.25) that $P^P(p) = \int_0^b \int_0^{\frac{D_b(1-F_b(p))}{\underline{F}_s^P(p)}} \frac{1}{bs} dD_s dD_b = \frac{b}{2s}\frac{1-p}{\underline{F}_s^P(p)}$. Case (2c). When $(\theta \leq) \tilde{p} \leq p \leq 1$, we have $\frac{s \cdot \tilde{F}_s^P(p)}{1-F_b(p)} \geq b$ and $\tilde{F}_s^P(p) = F_s(p)$. It follows that $P^P(p)$ has the same expression as in Case (1c).

Combining Cases (2a)-(2c), we have

$$p \cdot P^P(p) = \begin{cases} p\left[1 - \frac{s}{2b}\frac{\underline{F}_s^P(p)}{1-p}\right] & \text{if } 0 \leq p \leq \theta; \\ p\left[\frac{b}{2s}\frac{1-p}{\underline{F}_s^P(p)}\right] & \text{if } \theta \leq p \leq \tilde{p}; \\ p\left[\frac{b}{2s}\frac{1-p}{p}\right] & \text{if } \tilde{p} \leq p \leq 1. \end{cases} \tag{EC.29}$$

By comparing Eqn. (EC.29) and Eqn. (EC.23), we have $p \cdot P^P(p)$ for two cases, when $\underline{p} \leq \theta$ and when $\underline{p} \geq \theta$.

On one hand, when $\underline{p} \leq \theta (\leq \tilde{p})$, we have

$$p \cdot P^P(p) = \begin{cases} p\left[1 - \frac{s}{2b}\frac{\underline{F}_s^P(p)}{1-p}\right] = p & \text{if } 0 \leq p \leq \underline{p}; \\ p\left[1 - \frac{s}{2b}\frac{\underline{F}_s^P(p)}{1-p}\right] = \tilde{p} \cdot P^P(\tilde{p}) & \text{if } \underline{p} \leq p \leq \theta; \\ p\left[\frac{b}{2s}\frac{1-p}{\underline{F}_s^P(p)}\right] = \tilde{p} \cdot P^P(\tilde{p}) & \text{if } \theta \leq p \leq \tilde{p}; \\ p\left[\frac{b}{2s}\frac{1-p}{p}\right] & \text{if } \tilde{p} \leq p \leq 1. \end{cases} \tag{EC.30}$$

Eqn. (EC.30), together with Eqn. (EC.21) and the fact that no customer charges more than $\tilde{p}$ unless the reservation price constraint is binding, we have

$$\tilde{F}_s^P(p) = \begin{cases} 0 & \text{if } 0 \leq p \leq \underline{p}; \\ \frac{2b}{s}(1-p)(1 - \frac{\tilde{p} \cdot P^P(\tilde{p})}{p}) & \text{if } \underline{p} \leq p \leq \theta; \\ \frac{b}{2s}\frac{p(1-p)}{\tilde{p} \cdot P^P(\tilde{p})} & \text{if } \theta \leq p \leq \tilde{p}; \\ p & \text{if } \tilde{p} \leq p \leq 1. \end{cases} \tag{EC.31}$$

Because $\tilde{F}_s^P(p)$ needs to be continuous, we know that $\tilde{F}_s^P(\theta^-) = \tilde{F}_s^P(\theta^+) \Leftrightarrow \frac{2b}{s}(1-\theta)(1 - \frac{\tilde{p} \cdot P^P(\tilde{p})}{\theta}) = \frac{b}{2s}\frac{\theta(1-\theta)}{\tilde{p} \cdot P^P(\tilde{p})} \Leftrightarrow \theta = 2\tilde{p} \cdot P^P(\tilde{p})$. That is, $\tilde{F}_s^P(\theta) = \tilde{F}_s^P(\theta^-) = \tilde{F}_s^P(\theta^+) = \frac{b}{s}(1-\theta)$ and it is easy to verify that $b \cdot (1 - F_b(\theta)) = s \cdot \tilde{F}_s^P(\theta)$.

On the other hand, when $\underline{p} \geq \theta$, we have from Eqn. (EC.29) and Eqn. (EC.23) that

$$p \cdot P^P(p) = \begin{cases} p\left[1 - \frac{s}{2b}\frac{\underline{F}_s^P(p)}{1-p}\right] = p & \text{if } 0 \leq p \leq \theta; \\ p\left[\frac{b}{2s}\frac{1-p}{\underline{F}_s^P(p)}\right] = p & \text{if } \theta \leq p \leq \underline{p}; \\ p\left[\frac{b}{2s}\frac{1-p}{\underline{F}_s^P(p)}\right] = \tilde{p} \cdot P^P(\tilde{p}) & \text{if } \underline{p} \leq p \leq \tilde{p}; \\ p\left[\frac{b}{2s}\frac{1-p}{p}\right] & \text{if } \tilde{p} \leq p \leq 1. \end{cases} \tag{EC.32}$$

And it follows that

$$\tilde{F}_s^P(p) = \begin{cases} 0 & \text{if } 0 \le p \le \theta; \\ \frac{b}{2s}(1-p) & \text{if } \theta \le p \le \underline{p}; \\ \frac{b}{2s}\frac{p(1-p)}{\tilde{p}\cdot P^P(\tilde{p})} & \text{if } \underline{p} \le p \le \tilde{p}; \\ p & \text{if } \tilde{p} \le p \le 1. \end{cases} \tag{EC.33}$$

However, in this case, $\tilde{F}_s^P(p)$ is not continuous at $p = \underline{p}$, so there cannot be an equilibrium when $\underline{p} \ge \theta$. So for Case (2), the candidate $\tilde{p}$ satisfies $(\underline{p} \le)\theta \le \tilde{p}$.

To summarize all the analysis so far, from Case 1, we know that if $p^* \le \tilde{p} \le \theta$, $p \cdot P^P(p)$ is given by Eqn. (EC.27), and $\tilde{F}_s^P(p)$ is given by Eqn. (EC.28), also, $\tilde{p}\cdot P^P(\tilde{p})$ is decreasing in $\tilde{p}$. From Case 2, we know that if $\tilde{p} \ge \theta$, $p \cdot P^P(p)$ is given by Eqn. (EC.30), and $\tilde{F}_s^P(p)$ is given by Eqn. (EC.31), also $\tilde{p}\cdot P^P(\tilde{p})$ is decreasing in $\tilde{p}$. Therefore, combining Case 1 and Case 2, we know that when $\tilde{p} = p^* = 1 - \sqrt{\frac{s}{2b+s}}$, $\tilde{p}\cdot P^P(\tilde{p})$ achieves its maximum. Together with Eqn. (EC.22), we know that when $\tilde{p} = 1 - \sqrt{\frac{s}{2b+s}}$, all the sellers' expected revenue $p \cdot P^P(p)$ also achieve the maximum. That is, $\tilde{p} = 1 - \sqrt{\frac{s}{2b+s}}$ is the Pareto-dominant solution, with corresponding $\tilde{p}\cdot P^N(\tilde{p}) = \frac{b+s-\sqrt{(2b+s)s}}{b}$. Also, $p \cdot P^N(p)$ is given by Eqn. (EC.27), and $\tilde{F}_s^N(p)$ is given by Eqn. (EC.28).

Finally, we will establish that $\tilde{p} = 1 - \sqrt{\frac{s}{2b+s}}$ is indeed a Pareto-dominant equilibrium solution. In particular, we need to show that it satisfies $\int_0^1 \tilde{f}_s^P(p)dp = 1$ and $\tilde{F}_s^P(p) \le F_s(p)$.

From Eqn. (EC.28), we can deduce the probability density function, $\tilde{f}_s^P(p)$, as follows:

$$\tilde{f}_s^P(p) = \begin{cases} 0 & \text{if } 0 \le p \le \underline{p}; \\ \frac{2b}{s}(\frac{\tilde{p}\cdot P^P(\tilde{p})}{p^2} - 1) & \text{if } \underline{p} \le p \le \tilde{p}; \\ 1 & \text{if } \tilde{p} \le p \le 1. \end{cases} \tag{EC.34}$$

Because $\tilde{p} \le \theta$, we have $P^P(\tilde{p}) = 1 - \frac{s}{2b}\frac{\tilde{p}}{1-\tilde{p}}$. Thus,

$$\int_0^1 \tilde{f}_s^P(p)dp = \int_{\underline{p}=\tilde{p}\cdot P^P(\tilde{p})}^{\tilde{p}} \frac{2b}{s}(\frac{\tilde{p}\cdot P^P(\tilde{p})}{p^2} - 1)dp + \int_{\tilde{p}}^1 1dp = \frac{2b}{s}(1 - P^P(\tilde{p}))(1-\tilde{p}) + (1-\tilde{p}) = 1.$$

In what follows, we will show that, for any p, we have $\tilde{F}_s^P(p) \le F_s(p)$. From Eqn. (EC.28), we know that: 1. If $p \le \underline{p}$, we have $\tilde{F}_s^P(p) = 0 < p = F_s(p)$; 2. If $\underline{p} \le p \le \tilde{p}$, we have $\tilde{F}_s^P(p) - F_s(p) = \frac{2b}{s}(1-p)(1 - \frac{\tilde{p}\cdot P^P(\tilde{p})}{p}) - p$, where $\tilde{p}\cdot P^P(\tilde{p}) = \frac{b+s-\sqrt{s(2b+s)}}{b}$. Since the second-order derivative $\frac{\partial^2(\tilde{F}_s^P(p)-F_s(p))}{\partial p^2} = -\frac{4\left(b+s-\sqrt{s(2b+s)}\right)}{sp^3} \le 0$, then the first-order derivative $\frac{\partial(\tilde{F}_s^P(p)-F_s(p))}{\partial p} = \frac{2[b+s-\sqrt{s(2b+s)}]-p^2(2b+s)}{sp^2}$ is decreasing in p, that is $\frac{\partial(\tilde{F}_s^P(p)-F_s(p))}{\partial p} \ge \frac{\partial(\tilde{F}_s^P(p)-F_s(p))}{\partial p}|_{p=\tilde{p}} = 0$. Therefore, we can deduce that $\tilde{F}_s^P(p) - F_s(p)$ is increasing in p, then $\tilde{F}_s^P(p) - F_s(p) \le \tilde{F}_s^P(\tilde{p}) - F_s(\tilde{p}) = 0$, i.e., $\tilde{F}_s^P(p) \le F_s(p)$. 3. If $p \ge \tilde{p}$, we have $\tilde{F}_s^P(p) = p = F_s(p)$. □

Proof of Theorem 3 The sellers' listing price distribution $\tilde{F}_s^M(p|d_b, d_s)$ and selling probability function $P^M(p|d_b, d_s)$ can be deduced directly from the analysis in §5.2. □

Proof of Corollary 2 It is clear from (5) that $p_{max}^M(d_b, d_s)$ weakly increases in d_b and decreases in d_s. Therefore, we can immediately show that $\tilde{F}_s^M(p|d_b, d_s)$ weakly decreases in d_b and increases in d_s. That is, $Pr(p > x|d_b, d_s) = 1 - \tilde{F}_s^M(x|d_b, d_s)$ weakly increases in d_b and decreases in d_s for all $x \in [0, 1]$. □

Proof of Theorem 4 The sellers' listing price distribution $\tilde{F}_s^B(p|d_b, d_s)$ and selling probability function $P^B(p|d_b, d_s)$ can be deduced directly from the analysis in §5.3. □

Proof of Theorem 5   1. Compare Q in Model N and Model M: As $q^N = q^M$, we have $Q^N = Q^M$.

2. Compare Q in Model P and Model B: As in Model P, $\tilde{F}_s^P(p)$ stochastically dominates $F_s^P(p)$. Let us assume $p_{max}^B \geq p_{max}^P$, then $d_b[1-F_b(p_{max}^P)] = d_s \cdot \tilde{F}_s^P(p_{max}^P) \leq d_s \cdot F_s^P(p_{max}^P) \leq d_s \cdot F_s^P(p_{max}^B) = d_b[1-F_b(p_{max}^B)]$, which contradicts $d_b[1-F_b(p_{max}^P)] \geq d_b[1-F_b(p_{max}^B)]$. Hence, we have $p_{max}^B \leq p_{max}^P$. Therefore, we have $q^P = d_b[1-F_b(p_{max}^P)] \leq d_b[1-F_b(p_{max}^B)] = q^B$, which directly suggests that $Q^P \leq Q^B$.

3. Compare Q in Model M and Model B: Denote $\theta = \frac{b}{s}$, then

$$b(Q^M - Q^B)/s^2 = \begin{cases} \frac{1}{3}\theta^3 \log(1+1/\theta) + \frac{1}{3}\log(1+\theta) + \frac{1}{6}(-2\theta^3 + \theta^2 - 2\theta) & \text{if } \theta \leq \frac{1}{2}; \\ \frac{1}{3}\theta^3 \log(1+1/\theta) + \frac{1}{3}\log(1+\theta) - \frac{1}{24}(8\theta^2 + 2\theta + 1) & \text{if } \theta > \frac{1}{2}. \end{cases}$$

When $\theta \leq \frac{1}{2}$, we have $\frac{\partial(b(Q^M-Q^B)/s^2)}{\partial\theta} = \theta^2[\log\left(\frac{1}{\theta}+1\right) - 1] \geq \theta^2[\log(3)-1] > 0$, which implies that $b(Q^M - Q^B)/s^2$ increases in θ. In addition, when $\theta \to 0^+$, we have $b(Q^M - Q^B)/s^2 = \frac{1}{3}\theta^3\log(1+1/\theta) \to 0^+$. Therefore, $b(Q^M - Q^B)/s^2 \geq 0$. On the other hand, when $\theta > \frac{1}{2}$, we have $\frac{\partial^4(b(Q^M-Q^B)/s^2)}{\partial\theta^4} = -\frac{2}{\theta(\theta+1)^3} < 0$. Hence, $\frac{\partial^3(b(Q^M-Q^B)/s^2)}{\partial\theta^3}$ decreases in θ. When $\theta = 1/2$, we have $\frac{\partial^3(b(Q^M-Q^B)/s^2)}{\partial\theta^3} = \log(9) - \frac{16}{9} > 0$; and when $\theta \to +\infty$, we have $\frac{\partial^3(b(Q^M-Q^B)/s^2)}{\partial\theta^3} = 0$, which suggests $\frac{\partial^2(b(Q^M-Q^B)/s^2)}{\partial\theta^2}$ increases in θ. Similarly, when $\theta = 1/2$, we have $\frac{\partial^2(b(Q^M-Q^B)/s^2)}{\partial\theta^2} = \log(3) - \frac{4}{3} < 0$; and when $\theta \to +\infty$, we have $\frac{\partial^2(b(Q^M-Q^B)/s^2)}{\partial\theta^2} = 0$, which suggests that $\frac{\partial(b(Q^M-Q^B)/s^2)}{\partial\theta}$ decreases in θ. Next, when $\theta = 1/2$, we have $\frac{\partial(b(Q^M-Q^B)/s^2)}{\partial\theta} = \frac{1}{4}(\log(3)-1) > 0$; and when $\theta = 2$, we have $\frac{\partial(b(Q^M-Q^B)/s^2)}{\partial\theta} = \log\left(\frac{81}{16}\right) - \frac{7}{4} < 0$. That is, $b(Q^M - Q^B)/s^2$ first increases in θ and then decreases in θ. At last, when $\theta = 1/2$, we have $b(Q^M - Q^B)/s^2 = \frac{1}{24}\left(-4 + \log(3) + 8\log\left(\frac{3}{2}\right)\right) > 0$; and when $\theta = 2$, we have $b(Q^M - Q^B)/s^2 = -\frac{37}{24} - \frac{1}{3}8\log(2) + \log(27) < 0$. Therefore, there exists a threshold θ_1, whose numerical solution is approximately 1.06242, such that if $\frac{1}{2} \leq \theta \leq \theta_1$, then $Q^M \geq Q^B$; else if $\theta \geq \theta_1$, then $Q^M \leq Q^B$. Combining the case when $\theta \leq \frac{1}{2}$ and the case when $\theta > \frac{1}{2}$, we can conclude that if $\theta \leq \theta_1$, then $Q^M \geq Q^B$; else if $\theta \geq \theta_1$, then $Q^M \leq Q^B$. □

Proof of Proposition 2   Result follows from Theorem 5 and the analysis in §6.1. □

Proof of Theorem 6   We start with restating T in three models. From Proposition 1, we can state bsT^N explicitly as follows:

$$bsT^N = \begin{cases} \frac{1}{6}b^3[2 - \log(4b) + \log(s)] & \text{if } \frac{b}{s} \leq \frac{1}{2}; \\ \frac{1}{24}s(4b-s)[b\log\left(4 - \frac{s}{b}\right) - b\log(4) + s] & \text{if } \frac{b}{s} > \frac{1}{2}. \end{cases}$$

For Model M: Recall that we have $t^M = \frac{d_b^2}{d_s}$ for $d_b \leq \frac{d_s}{2}$ and $t^M = \frac{d_s}{4}$ for $d_b \geq \frac{d_s}{2}$. Directly, we can state bsT^M as follows:

$$bsT^M = \begin{cases} \frac{1}{3}b^3\log\left(\frac{s}{b}\right) + \frac{1}{18}b^3(5 - \log(64)) & \text{if } \frac{b}{s} \leq \frac{1}{2}; \\ \frac{1}{72}s^2(9b - 2s) & \text{if } \frac{b}{s} \geq \frac{1}{2}. \end{cases}$$

For Model B: From $t^B = \frac{d_b^2 \cdot d_s}{(d_b+d_s)^2}$, we have

$$bsT^B = \frac{1}{3}b^3\log\left(\frac{b+s}{b}\right) - \frac{2}{3}s^3\log\left(\frac{b+s}{s}\right) + \frac{1}{3}bs(2s-b).$$

Next, we will compare T in three models:

1. Compare T in Model N and Model M: Let's define $\theta = \frac{b}{s}$. Then $b(T^N - T^M)/s^2$ is given as follows:

$$b(T^N - T^M)/s^2 = \begin{cases} \frac{1}{18}\theta^3(3\log(\theta) + 1) & \text{if } \theta \leq \frac{1}{2}; \\ \frac{1}{72}\left(-12\theta^2\log(4) + 3\theta(1 + \log(4)) + 3(4\theta - 1)\theta\log\left(4 - \frac{1}{\theta}\right) - 1\right) & \text{if } \theta > \frac{1}{2}. \end{cases}$$

When $\theta \le \frac{1}{2}$, we have, $\frac{\partial(b(T^N-T^M)/s^2)}{\partial\theta} = \frac{1}{6}\theta^2(3\log(\theta)+2) \le 0$. In addition, we have $b(T^N-T^M)/s^2 = 0$ for $\theta \to 0^+$ and $b(T^N-T^M)/s^2 = \frac{1}{144}(1-\log(8)) < 0$ for $\theta = 1/2$. Therefore, for $\theta \le \frac{1}{2}$, we have $T^N \le T^M$. When $\theta \ge \frac{1}{2}$, we have $\frac{\partial(b(T^N-T^M)/s^2)}{\partial\theta} = \frac{\left(-8\theta\log(4)+(8\theta-1)\log\left(4-\frac{1}{\theta}\right)+2+\log(4)\right)}{24} \le 0$. Together with the fact that $b(T^N-T^M)/s^2 = \frac{1}{144}(1-\log(8)) < 0$ for $\theta = 1/2$ and $T^N \le T^M$ for $\theta \ge \frac{1}{2}$. Combining these two cases, we conclude that $T^N \le T^M$.

2. Compare T in Model M and Model B: We can state $b(T^M-T^B)/s^2$ as follows:

$$b(T^M-T^B)/s^2 = \begin{cases} \frac{1}{3}\left(2-\theta^3\right)\log\left(1+\theta\right)+\frac{1}{18}\left(5\theta^3-6\theta^3\log(2)+6\theta^2-12\theta\right) & \text{if } \theta \le \frac{1}{2}; \\ -\frac{1}{3}\theta^3\log\left(1+1/\theta\right)+\frac{2}{3}\log\left(1+\theta\right)-\frac{1}{72}\left(-24\theta^2+39\theta+2\right) & \text{if } \theta > \frac{1}{2}. \end{cases}$$

When $\theta \le \frac{1}{2}$, we can show that $\frac{\partial^4(b(T^M-T^B)/s^2)}{\partial\theta^4} = -\frac{2(\theta(\theta(\theta+4)+6)+6)}{(\theta+1)^4} < 0$. Together with the observations that $\frac{\partial^3(b(T^M-T^B)/s^2)}{\partial\theta^3} = 3-\log(4) > 0$ for $\theta \to 0^+$ and $\frac{\partial^3(b(T^M-T^B)/s^2)}{\partial\theta^3} = \frac{10}{27}-\log(9) < 0$ for $\theta = 1/2$, we have that $\frac{\partial^2(b(T^M-T^B)/s^2)}{\partial\theta^2}$ first increases in θ and then decreases in θ. Similarly, we can show that $\frac{\partial(b(T^M-T^B)/s^2)}{\partial\theta}$ first increases in θ and then decreases in θ; and $b(T^M-T^B)/s^2$ increases in θ. We can further show that $b(T^M-T^B)/s^2 \to 0^+$ for $\theta \to 0^+$ and $b(T^M-T^B)/s^2 = \log\left(\frac{3^{5/8}}{2^{2/3}}\right) - \frac{31}{144} > 0$ for $\theta = 1/2$. Therefore, in the case of $\theta \le \frac{1}{2}$, we have $b(T^M-T^B)/s^2 > 0$, i.e., $T^M \ge T^B$. For $\theta \ge \frac{1}{2}$, we can show that $\frac{\partial^4(b(T^M-T^B)/s^2)}{\partial\theta^4} = -\frac{4\theta-2}{\theta(\theta+1)^4} < 0$. Together with the two boundary conditions that $\frac{\partial^3(b(T^M-T^B)/s^2)}{\partial\theta^3} = \frac{64}{27}-\log(9) > 0$ for $\theta = 1/2$ and $\frac{\partial^3(b(T^M-T^B)/s^2)}{\partial\theta^3} = 0$ for $\theta \to +\infty$, we conclude that $\frac{\partial^2(b(T^M-T^B)/s^2)}{\partial\theta^2}$ increases in θ. Similarly, we can show that $\frac{\partial(b(T^M-T^B)/s^2)}{\partial\theta}$ decreases in θ. Next, note that $\frac{\partial(b(T^M-T^B)/s^2)}{\partial\theta} = \frac{1}{24}(7-6\log(3)) > 0$ for $\theta = 1/2$ and $\frac{\partial(b(T^M-T^B)/s^2)}{\partial\theta} = \frac{35}{24} - 4\log\left(\frac{3}{2}\right) < 0$ for $\theta = 2$, which suggest that $b(T^M-T^B)/s^2$ first increases in θ and then decreases in θ. At last, Since we have $b(T^M-T^B)/s^2 = \log\left(\frac{3^{5/8}}{2^{2/3}}\right) - \frac{31}{144} > 0$ for $\theta = 1/2$ and $b(T^M-T^B)/s^2 = \frac{2}{9} + \log\left(\frac{4\cdot 2^{2/3}}{9}\right) < 0$ for $\theta = 2$, there exists a threshold θ_2, whose numerical solution is approximately 0.919008, such that if $\frac{1}{2} \le \theta \le \theta_2$, then $T^M \ge T^B$; else if $\theta \ge \theta_2$, then $T^M \le T^B$.

3. Compare T in Model N and Model B: We first state $b(T^N-T^B)/s^2$ as follows:

$$b(T^N-T^B)/s^2 = \begin{cases} \frac{1}{6}\theta^3(2-\log(4\theta)) + \frac{1}{3}\left(\theta^3\log\left(\frac{\theta}{\theta+1}\right) + (\theta-2)\theta + 2\log(\theta+1)\right) & \text{if } \theta \le \frac{1}{2}; \\ \frac{1}{3}\left(\theta^3\log\left(\frac{\theta}{\theta+1}\right) + \theta(\theta-2) + 2\log(\theta+1)\right) - \frac{1}{24}(4\theta-1)\left(\theta\log\left(\frac{4\theta}{4\theta-1}\right)-1\right) & \text{if } \theta > \frac{1}{2}. \end{cases}$$

When $\theta \le \frac{1}{2}$, we have $\frac{\partial(b(T^N-T^B)/s^2)}{\partial\theta} = \frac{\theta^2\left(5\theta-3(\theta+1)\log(4\theta)+6(\theta+1)\log\left(\frac{\theta}{\theta+1}\right)+11\right)}{6(\theta+1)}$. Since we have $\frac{\partial(b(T^N-T^B)/s^2)}{\partial\theta} \to 0^+$ for $\theta \to 0^+$ and $\frac{\partial(b(T^N-T^B)/s^2)}{\partial\theta} = \frac{1}{8}(3-\log(18)) \ge 0$ for $\theta = 1/2$, $b(T^N-T^B)/s^2$ first decreases in θ and then increases in θ. As $b(T^N-T^B)/s^2 \to 0^+$ for $\theta \to 0^+$ and $b(T^N-T^B)/s^2 = \frac{1}{48}(-10-33\log(2)+30\log(3)) \ge 0$ for $\theta = 1/2$, there exists a threshold θ_3, whose numerical solution is approximately 0.323018, such that if $0 \le \theta \le \theta_3$, then $T^N \le T^B$; else if $\theta_3 \le \theta \le \frac{1}{2}$, then $T^N \ge T^B$. When $\theta \ge \frac{1}{2}$, we can show that $\frac{\partial^2(b(T^N-T^B)/s^2)}{\partial\theta^2} = \frac{\theta(\theta(8\theta(24\theta+31)-57)+6)+8\theta(4\theta-1)(\theta+1)^2(-\log(-4\theta)+6\theta(\log(\theta)-\log(\theta+1))+\log(1-4\theta))-1}{24\theta(\theta+1)^2(4\theta-1)} < 0$. Since $\frac{\partial(b(T^N-T^B)/s^2)}{\partial\theta} = \frac{1}{8}(3-\log(18)) > 0$ for $\theta = 1/2$ and $\frac{\partial(b(T^N-T^B)/s^2)}{\partial\theta} = \frac{1}{24}(17-38\log(2)+7\log(3)) < 0$ for $\theta = 1$, we conclude that $b(T^N-T^B)/s^2$ first increases in θ and then decreases in θ. Finally, we observe that $b(T^N-T^B)/s^2 = \frac{1}{48}(-10-33\log(2)+30\log(3)) > 0$ for $\theta = 1/2$ and $b(T^N-T^B)/s^2 = \frac{1}{24}(\log(108)-5) < 0$ for $\theta = 1$. Therefore, there exists a threshold θ_4, whose numerical solution is approximately 0.730385, such that if $\frac{1}{2} \le \theta \le \theta_4$, then $T^N \ge T^B$; else if $\theta \ge \theta_4$, then $T^N \le T^B$. $\square$

Proof of Proposition 3 Result follows from Theorem 6 and the analysis in §6.2. $\square$